

# $G$ -LINKS INVARIANTS, MARKOV TRACES AND THE SEMI-CYCLIC $U_q\mathfrak{sl}(2)$ -MODULES.

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ABSTRACT. Kashaev and Reshetikhin proposed a generalization of the Reshetikhin-Turaev link invariant construction to tangles with a flat connection in a principal  $G$ -bundle of the complement of the tangle. The purpose of this paper is to adapt and renormalize their construction to define invariants of  $G$ -links using the semi-cyclic representations of the non-restricted quantum group associated to  $\mathfrak{sl}(2)$ , defined by De Concini and Kac. Our construction uses a modified Markov trace. In our main example, the semi-cyclic invariants are a natural extension of the generalized Alexander polynomial invariants defined by Akutsu, Deguchi, and Ohtsuki. Surprisingly, direct computations suggest that these invariants are actually equal.

## 1. INTRODUCTION

1.1. A major achievement in quantum topology was the construction of invariants of links and tangles using representations of a quantum group and more generally ribbon categories, due to Reshetikhin and Turaev in [17]. Turaev gave a theoretical generalization of this construction [18, 19] to  $G$ -links and  $G$ -tangles: tangles with a flat connection in a principal  $G$ -bundle over the complement of the tangle. The construction is parallel to [17] and leads to HQFT and invariants of 3-manifolds with a representation of their fundamental group in  $G$ . The algebraic data used in this theory is a ribbon  $G$ -category which in particular is a  $G$ -graded tensor category equipped with a homotopy braiding. It is an open problem to find interesting examples of ribbon  $G$ -categories where  $G$  is an infinite non-abelian group.

In [14], Kashaev and Reshetikhin propose a modification of Turaev's construction: for a group  $G$  having a factorization  $G = G_+G_-$ , they introduce the group  $G^* = G_+ \times G_-$  and define a notion of a  $G^*$ -ribbon category which is slightly different from the previous one. In particular, a  $G^*$ -ribbon category  $\mathcal{C}$  is  $G^*$ -graded category which is equipped with what could be called a *holonomy braiding*. Loosely speaking, a holonomy braiding is a pair  $(\mathcal{R}, c)$  of maps described as follows. First, the factorization of  $G$  produces a map  $\mathcal{R} : G^* \times G^* \rightarrow G^* \times G^*$  satisfying the set-theoretical Yang-Baxter equation. Second,  $c$  is a natural isomorphism which assigns to each pair of objects  $(V, W)$  of  $\mathcal{C}$  an isomorphism  $V \otimes W \rightarrow W' \otimes V'$  where  $(V', W')$  depends functorially on  $(V, W)$  and  $c$  is a lift of  $\mathcal{R}$  to  $\mathcal{C} \times \mathcal{C}$ . Given a  $G$ -ribbon category  $\mathcal{C}$  and a kind of section  $G^* \rightarrow \text{Obj}(\mathcal{C})$  the construction of [14] produces a  $G$ -link invariant. As stated in the conclusion of [14] the work of Kashaev and Reshetikhin was motivated by the category of representation of the unrestricted quantum groups at root of unity, in particular in the case of quantum  $\mathfrak{gl}_2$ . They show that even if this quantum group does not have an  $R$ -matrix, the conjugation by the  $R$ -matrix still induces a (not inner) automorphism of  $U_q\mathfrak{gl}_2^{\otimes 2}$ . The point is that this automorphism induces a map  $\mathcal{R}$  where the group  $G^*$  is obtained from the Poisson-Lie groups factorization of  $GL_2$  (see [20]).

The work of this paper is the beginning of a project to show that  $G$ -link invariant lead to invariants of 3-manifolds. In this context the motivating example comes from the representation theory over a non-restricted quantum group at root of unity associated to any simple Lie algebra

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$\mathfrak{g}$ , defined and studied by De Concini, Kac and Procesi in [6, 7, 8]. The first step in obtaining such a 3-manifold invariant is to construct a link invariant in the setting of these non-restricted quantum groups. In attempting to do this we ran into several problems: First, the factorization  $G = G_+ G_-$  is not strict in this context as  $G_+ \cap G_-$  is not trivial. Second, we were not able to define a holonomy braiding for every pair of objects in the category. Third, we had problems checking the compatibility of the partially defined holonomy braiding with the duality morphisms. Finally, the quantum dimension of a generic  $U_\xi(\mathfrak{g})$ -module vanishes which implies that the corresponding  $G$ -link invariant is trivial.

In this paper we show that these problems can be overcome when  $G$  is the Borel of  $SL_2(\mathbb{C})$  and  $\mathcal{C}$  is the category of the so called semi-cyclic representations of  $U_\xi(\mathfrak{sl}(2))$ . To do this, we first slightly generalize the definition of a factorization of a group to fit this example. Then we use a section  $G^* \rightarrow \mathcal{C}$  to show that it is enough to consider a holonomy braiding defined on a generic set of modules (this observation is already present in the work of [14]). To avoid the third problem, we work with braids and a  $G$ -link generalization of Markov traces. Finally, we re-normalize the invariants to overcome the fourth problem.

The algebraic structures in the main example of this paper are related to the work of Baseilhac in [3] where he considers a bundle over  $G$  of irreducible representation of  $U_\xi(\mathfrak{sl}(2))$  which is relevant in 3-dimensional quantum hyperbolic field theories.

In [1], Akutsu, Deguchi, and Ohtsuki defined generalized Alexander polynomial invariants of links. These generalized invariants can naturally be interpreted as maps on the space of abelian representations of the fundamental group into  $SL_2(\mathbb{C})$ , see [5]. We call these invariants the *nilpotent-ADO* invariants. The invariants of  $G$ -links defined in this paper can be seen as the natural extension of these maps to the space of reducible representations of the fundamental group into  $SL_2(\mathbb{C})$  (a representation  $\rho$  is reducible if there is line in  $\mathbb{C}^2$  which is invariant under the action of the image of  $\rho$ ). Surprisingly, experimental computations seem to indicate that this extension is trivial. In particular, the abelianization of a representation of the fundamental group into the Borel of  $SL_2(\mathbb{C})$  is a cohomology class in  $H^1(S^3 \setminus L, \mathbb{C}^*)$ . Then all of our computations suggest that the invariants of this paper only depend on this abelianization and so they seem to be equal to their corresponding nilpotent-ADO invariants.

The paper is organized as follows. In Section 3 we give the notions and some properties of factorized groups,  $G$ -links and  $G^*$ -braids. In Section 4 we introduce  $G^*$ -Markov trace and trace coloring systems. In Section 5 we recall some basic results on modified traces, which are the technical tools underlying our  $G^*$ -Markov traces. Section 6 is devoted to the example of the category of semi-cyclic representations of  $U_\xi(\mathfrak{sl}(2))$ .

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## 2. OUTLINE OF HOW TO DEFINE THE INVARIANT

In this section we give a short outline of how to use the general results of this paper. In Section 6 we will apply these results to the setting of  $U_\xi(\mathfrak{sl}(2))$ . The terms used in this section will be defined rigorously in subsequent sections.

We start with a  $\mathbb{k}$ -linear tensor category  $\mathcal{C}$  which is related to a factorized group  $(G, \overline{G}, G^*)$  as follows. Let  $Y \subset G$  be a union of conjugacy classes in  $\overline{G}$  and  $Y^*$  be the corresponding set in  $G^*$ . Then there exists a bijection  $\mathcal{R} : Y^* \times Y^* \rightarrow Y^* \times Y^*$  given by  $(x, y) \mapsto (x_L(x, y), x_R(x, y))$  which satisfies the Yang-Baxter equation. Here the functions  $x_L$  and  $x_R$  are defined using the

factorized group structure. We require that the map  $\mathcal{R}$  is compatible with a holonomy braiding: the category  $\mathcal{C}$  has a family of objects  $\mathbf{A} = \{A_y\}_{y \in Y^*}$  and invertible morphisms

$$\{C_{y,z} : A_y \otimes A_z \rightarrow A_{x_R(y,z)} \otimes A_{x_L(y,z)}\}_{y,z \in Y^*}$$

satisfying the braid relations.

A  $Y$ -admissible  $G$ -link is a framed oriented link  $L$  in  $S^3$  equipped with a group homomorphism  $\rho : \pi_1(S^3 \setminus L, \infty) \rightarrow G$  such that  $\rho(m) \in Y$  for any meridian  $m$  of any component of  $L$ . If  $\sigma$  is a braid with  $n$  strands whose closure is  $L$  then Proposition 3.3 implies that  $\rho$  induces a natural  $G^*$ -coloring on  $\sigma$ , i.e. a certain element  $\underline{y} \in (Y^*)^n$ . Then in the usual way the holonomy braiding and the braid generators induces a map

$$f_{(\sigma, \underline{y})} : A_{y_1} \otimes A_{y_2} \otimes \cdots \otimes A_{y_n} \rightarrow A_{y_1} \otimes A_{y_2} \otimes \cdots \otimes A_{y_n}.$$

The final requirement on  $\mathcal{C}$  is the existence of a  $G^*$ -Markov trace which can be thought of as a family of twists  $\{\theta_y\}_{y \in Y^*}$  and a map  $t : \{f_{(\sigma, \underline{y})}\} \rightarrow \mathbb{k}$ , where  $\{(\sigma, \underline{y})\}$  are indexed by all  $G^*$ -colored braids which correspond to some  $Y$ -admissible  $G$ -link. The  $G^*$ -Markov trace satisfies some relations, including

$$(1) \quad t\left(f_{(\sigma\sigma', \underline{y})}\right) = t\left(f_{(\sigma', \underline{y}')} \right) \quad \text{and} \quad t\left(f_{(\sigma_n^{\pm 1} \sigma, \underline{y}'')}\right) = (\theta_y)^{\pm 1} t\left(f_{(\sigma, \underline{y})}\right)$$

where  $\sigma_n$  is the  $n^{\text{th}}$  generator of braid group on  $n+1$  strands and  $\underline{y}'$  and  $\underline{y}''$  are certain tuples of elements of  $Y^*$  determined by  $(\sigma, \underline{y})$ . If  $(\sigma, \underline{y})$  is any  $G^*$ -colored braid corresponding to a  $Y$ -admissible  $G$ -link  $L$ , then Theorem 4.3 states that  $F'(L) = t\left(f_{(\sigma, \underline{y})}\right)$  is a well defined invariant of  $L$ . The proof of the theorem is essentially that there exists a finite number of  $G^*$ -Markov moves relating any two  $G^*$ -colored braids  $(\sigma, \underline{y})$  and  $(\sigma', \underline{y}')$  whose closures are isotopic to the same  $G$ -link. Then the relations of Equation (1) imply that the  $G^*$ -Markov trace preserves these  $G^*$ -Markov moves and so leads to a well defined invariant.

### 3. $G$ -LINKS AND $G^*$ -BRAIDS

**3.1.  $G$ -links and factorized groups.** Let  $G$  be a group. A  $G$ -link is a framed oriented link  $L$  in  $S^3$  and a group homomorphism  $\rho : \pi_1(S^3 \setminus L, \infty) \rightarrow G$ . We say  $\rho$  is a representation on  $L$ . If  $Y$  is a union of conjugacy classes in  $\overline{G}$  of elements of  $G$ , we say that  $\rho$  is a  $Y$ -admissible representation if  $\rho(m) \in Y$  for any meridian  $m$  of any component of  $L$ .

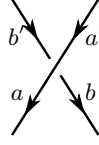
Given a planar projection of  $L$ , the map  $\rho$  can be extended to a map on the meridian of the edges of the planar diagram. This extension is canonical in the context of a factorized group and leads to a description of  $G$ -links as colored braids. To do this we will now define the notion of a *factorized group* which is a tuple  $(G, \overline{G}, G^*, \varphi_+, \varphi_-)$  where:

- (1)  $G, \overline{G}$  and  $G^*$  are groups such that  $G$  is a normal subgroup of  $\overline{G}$ ,
- (2) there exists group morphisms  $\varphi_+, \varphi_- : G^* \rightarrow \overline{G}$  such that the map

$$\psi : G^* \rightarrow \overline{G} \quad \text{given by} \quad x \mapsto \varphi_+(x)\varphi_-(x)^{-1}$$

realize a bijection between  $G^*$  and  $G$ .

This notion of a factorized group is a slight generalization of the treatment of factorizable groups given in [14]. If  $x \in G$ , we denote  $x_{\pm} = \varphi_{\pm}(\psi^{-1}(x)) \in \overline{G}$ . So we have  $x = x_+x_-^{-1}$ . The map  $\psi$  is not a group morphism but if  $x = \psi(x')$ ,  $y = \psi(y')$  then  $\psi(x'y') = x_-x_yx_-^{-1}$  and  $\psi(x'^{-1}) = x_-^{-1}x^{-1}x_-$ .

FIGURE 1. The edges at a crossing of  $D$ 

**3.2.  $G^*$ -diagrams.** Let  $(G, \overline{G}, G^*, \varphi_+, \varphi_-)$  be a factorized group. Let  $D$  be a regular planar projection of a framed oriented link  $L$ . Then  $D$  is a quadrivalent graph embedded in  $\mathbb{R}^2$  with the “over-crossing information” at its vertices. Let  $\mathcal{E}_D$  be the set of oriented edges of  $D$ . A  $G^*$ -coloring of  $D$  is a map  $\rho^* : \mathcal{E}_D \rightarrow G^*$  that satisfies

- for  $e \in \mathcal{E}_D$ , we have  $\rho^*(-e) = \rho^*(e)^{-1}$  where  $-e$  is the edge  $e$  with opposite orientation,
- for any vertex  $v$  of  $D$ , let  $a, b, a', b'$  be the four edges adjacent to  $v$  as shown in Figure 1, if  $x = \rho^*(a), y = \rho^*(b), x' = \rho^*(a'), y' = \rho^*(b') \in G$  then we have

$$(2) \quad \begin{cases} y' x' & = & x y \in G^* \\ \varphi_+(y') \varphi_-(x') & = & \varphi_-(x) \varphi_+(y) \in \overline{G}. \end{cases}$$

**Proposition 3.1** (see also [14]). *There is a natural bijection between the set of  $G$ -valued representations on  $L$  and the set of  $G^*$ -colorings of  $D$ :*

$$\{\rho : \pi_1(S^3 \setminus L, \infty) \rightarrow G\} \simeq \{\rho^* : \mathcal{E}_D \rightarrow G^*\}.$$

Furthermore, if  $Y \subset G$  is a union of conjugacy classes in  $\overline{G}$  and  $Y^* = \psi^{-1}(Y)$  then this bijection identifies  $Y$ -admissible representations with  $G^*$ -colorings with values in  $Y^*$ .

We will prove Proposition 3.1 in Section 3.5.

**3.3.  $G^*$ -braids.** Let  $H$  be a set and let  $\mathcal{R} : H \times H \rightarrow H \times H$  be a bijective map that satisfies the set-theoretical Yang-Baxter equation:

$$(3) \quad \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} = \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23}$$

here all mappings are from  $(H)^{\times 3}$  to itself and  $\mathcal{R}_{ij}$  is the usual mapping (e.g. the mapping  $\mathcal{R}_{12}$  acts as  $\mathcal{R}$  in the first two factors and trivially in the last one).

Let  $\tau : H \times H \rightarrow H \times H$  be the flip map defined by  $(y, y') \mapsto (y', y)$ . For  $n \geq 1$ , let  $\{\sigma_i\}_{i=1, \dots, n-1}$  be the usual generators of the braid group  $B_n$  on  $n$  strands. Let  $\text{Bij}(H^n)$  be the group of bijections of  $H^n$ . Consider the group homomorphism  $B_n \rightarrow \text{Bij}(H^n)$ ,  $\sigma \mapsto \sigma_{\mathcal{R}}$  induced by the assignment  $\sigma_i \mapsto (\tau \circ \mathcal{R})_{i, i+1}$  where  $(\tau \circ \mathcal{R})_{i, i+1}$  is the bijection of  $H^n$  given by  $(\tau \circ \mathcal{R})$  in the  $i^{\text{th}}$  and  $(i+1)^{\text{th}}$  slots and  $\text{Id}_H$  otherwise. A  $H$ -coloring of the braid  $\sigma$  is a fixed point of this action, i.e. an element  $\underline{y} = (y_1, \dots, y_n) \in H^n$  such that  $\sigma_{\mathcal{R}}(\underline{y}) = \underline{y}$ .

We will now show that a factorized group gives rise to a natural Yang-Baxter map  $\mathcal{R}$ . Let  $(G, \overline{G}, G^*, \varphi_+, \varphi_-)$  be a factorized group. As before, let  $Y \subset G$  be a union of conjugacy classes in  $\overline{G}$  and  $Y^* = \psi^{-1}(Y)$ . Let  $\times_R, \times_L : G^* \times G^* \rightarrow G^*$  be the maps defined by

$$\times_R(x, y) = \psi^{-1}(\varphi_-(x) \psi(y) \varphi_-(x)^{-1}) \quad \text{and} \quad \times_L(x, y) = \psi^{-1}(\varphi_+(\times_R(x, y))^{-1} \psi(x) \varphi_+(\times_R(x, y))).$$

**Proposition 3.2.** *The map  $\mathcal{R} : G^* \times G^* \rightarrow G^* \times G^*$  defined by  $(x, y) \mapsto (\times_L(x, y), \times_R(x, y))$  is bijective and satisfies the Yang-Baxter Equation (3). Moreover, it restricts to a bijection  $\mathcal{R} : Y^* \times Y^* \rightarrow Y^* \times Y^*$ .*

We will prove Proposition 3.2 in Section 3.5.

**Proposition 3.3.** *Let  $\sigma \in B_n$  be a braid whose braid closure is isotopic to a framed oriented link  $L$ . There is a natural bijection between the set of  $G$ -valued representations on  $L$  and the set of  $G^*$ -colorings of  $\sigma$ :*

$$\{\rho : \pi_1(S^3 \setminus L, \infty) \rightarrow G\} \simeq \{\underline{y} \in (G^*)^n \mid \sigma_{\mathcal{R}}(\underline{y}) = \underline{y}\}.$$

Furthermore, this bijection identifies  $\mathbf{Y}$ -admissible representation with  $\mathbf{Y}^*$ -coloring of  $\sigma$ .

We will prove Proposition 3.3 in Section 3.5.

**3.4. Example.** In this paper we will work with the following factorized group. Let  $\overline{G}$  be the group of  $2 \times 2$  upper triangular invertible matrices and  $G = \overline{G} \cap \mathrm{SL}_2(\mathbb{C})$ . Let

$$G^* = \left\{ \begin{pmatrix} 1 & \varepsilon \\ 0 & \kappa \end{pmatrix} : \kappa \in \mathbb{C}^*, \varepsilon \in \mathbb{C} \right\}$$

and let  $\varphi_- : G^* \rightarrow \overline{G}$  be the inclusion map. Finally, let  $\varphi_+ : G^* \rightarrow \overline{G}$  be the map given by

$$\varphi_+ \left( \begin{pmatrix} 1 & \varepsilon \\ 0 & \kappa \end{pmatrix} \right) = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we have factorized the upper triangular matrices of  $\mathrm{SL}_2(\mathbb{C})$  as

$$\begin{pmatrix} \kappa^{-1} & \varepsilon \\ 0 & \kappa \end{pmatrix} = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -\varepsilon \\ 0 & \kappa \end{pmatrix}^{-1}.$$

Then Proposition 3.2 gives a Yang-Baxter map:

$$\mathcal{R} \left( \begin{pmatrix} 1 & \varepsilon_1 \\ 0 & \kappa_1 \end{pmatrix}, \begin{pmatrix} 1 & \varepsilon_2 \\ 0 & \kappa_2 \end{pmatrix} \right) = \left( \begin{pmatrix} 1 & \varepsilon_1/\kappa_2 \\ 0 & \kappa_1 \end{pmatrix}, \begin{pmatrix} 1 & (\varepsilon_2 + \varepsilon_1(\kappa_2 - \kappa_2^{-1}))/\kappa_1 \\ 0 & \kappa_2 \end{pmatrix} \right).$$

In this example, we define  $\mathbf{Y}^* = \left\{ \begin{pmatrix} 1 & \varepsilon \\ 0 & \kappa \end{pmatrix} : \kappa \in \mathbb{C}^* \setminus \{\pm 1\}, \varepsilon \in \mathbb{C} \right\}$ . Then  $\mathbf{Y} = \psi(\mathbf{Y}^*)$  is the set of matrices in  $G$  whose trace is not  $\pm 2$  and so  $\mathbf{Y}$  is a union of conjugacy classes in  $\overline{G}$ .

**3.5. Proofs of propositions.** In this subsection we will prove Propositions 3.1, 3.2 and 3.3 using the ideas of [14]. To do this we first need to develop some terminology.

Let  $L$  be a framed oriented link  $L$  in  $S^3 = \mathbb{R}^3 \cup \infty$ . Let  $D \subset \mathbb{R}^2 \times \{0\}$  be a regular planar projection of  $L$ . The diagram  $D$  splits  $\mathbb{R}^2 \times \{0\}$  into regions  $R_\infty, R_1, \dots, R_n$  (where we assume that  $R_\infty$  is the infinite region). For each choose a point  $M_i \times \{0\}$  in  $R_i \subset \mathbb{R}^2 \times \{0\}$ . Then the fundamental group  $\pi_1(S^3 \setminus L, \infty)$  is generated by the downward oriented vertical lines  $\vec{R}_i = M_i \times \mathbb{R}$ , which are loops through  $\infty$ . Here  $\vec{R}_\infty$  is the trivial loop. Split the loop  $\vec{R}_i$  into two paths  $\vec{R}_i^- = M_i \times \mathbb{R}^-$  and  $\vec{R}_i^+ = M_i \times \mathbb{R}^+$  both oriented from  $\infty$  to  $M_i$  (see Figure 2(A)). In the groupoid  $\bar{\pi} = \pi_1(S^3 \setminus L, \{M_i\})$ , we have  $\vec{R}_i = \vec{R}_i^+ \overleftarrow{R}_i^-$  (where  $\overleftarrow{X}$  is defined to be the path  $\vec{X}$  in the opposite direction and we use the contravariant concatenation of paths in which  $\gamma\delta$  is defined when  $\gamma(1) = \delta(0)$ ).

As above, let  $\mathcal{E}_D$  be the set of oriented edges of  $D$ . If  $e \in \mathcal{E}_D$  is an oriented edge with regions  $R_i$  on the right and  $R_j$  on the left, we define the meridian of  $e$  as the loop based at  $M_i$  given by  $m_e = \overleftarrow{R}_i^+ \vec{R}_j^+ \overleftarrow{R}_j^- \vec{R}_i^- \in \bar{\pi}$ , see Figure 2(B). (Note the orientation of  $m_e$  is opposite to the one induced by  $e$ .) We have  $m_e = m_e^+(m_e^-)^{-1}$  where  $m_e^+ = \overleftarrow{R}_i^+ \vec{R}_j^+$  and  $m_e^- = \overleftarrow{R}_i^- \vec{R}_j^-$ . In  $\bar{\pi}$ , also we have  $m_e = \overleftarrow{R}_i^+ \vec{R}_j \overleftarrow{R}_i \vec{R}_i^+$  which is conjugate to the loop  $\vec{R}_j \overleftarrow{R}_i$  based at  $\infty$ .

The following lemma can be deduced from the classical presentation of  $\pi_1(S^3 \setminus L, \infty)$  associated to the diagram  $D$ .

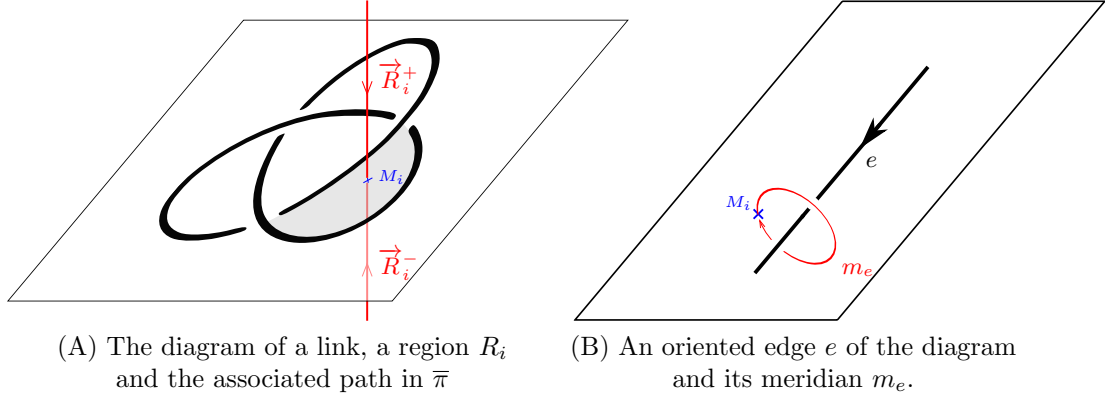


FIGURE 2. Diagrams of a link

**Lemma 3.4.** *The groupoid  $\bar{\pi} = \pi_1(S^3 \setminus L, \{M_i\})$  is generated by the elements  $\{m_e^+, m_e^- : e \in \mathcal{E}_D\}$  subject to the following relations :*

- For any  $e \in \mathcal{E}_D$ , if  $-e$  is the same edge with opposite orientation, then  $m_{-e}^\pm = (m_e^\pm)^{-1}$ .
- For any vertex  $v$  of  $D$ , if  $a, b, a', b'$  are the four edges adjacent to  $v$  as shown in Figure 1 then one has

$$(4) \quad \begin{cases} m_{b'}^+ m_{a'}^+ &= m_a^+ m_b^+ \\ m_{b'}^- m_{a'}^- &= m_a^- m_b^- \\ m_{b'}^+ m_{a'}^- &= m_a^- m_b^+ \end{cases}$$

*Proof of Proposition 3.1.* Let  $\rho : \pi_1(S^3 \setminus L, \infty) \rightarrow G$  be a representation on  $L$ . Then  $\rho$  can be extended to a map of groupoids  $\bar{\rho} : \bar{\pi} \rightarrow \bar{G}$  by setting  $\bar{\rho}(\vec{R}_i^\pm) = \rho(\vec{R}_i)_\pm$ . In particular, we have  $\bar{\rho}(\vec{R}_i) = \rho(R_i) \in G$  and  $\bar{\rho}(m_e) = \bar{\rho}(\vec{R}_i^+)^{-1} \rho(\vec{R}_j \overleftarrow{R}_i) \bar{\rho}(\vec{R}_i^+) \in \bar{\rho}(\vec{R}_i^+)^{-1} G \bar{\rho}(\vec{R}_i^+) = G$ . We define a map  $\rho^* : \mathcal{E}_D \rightarrow G^*$  that expresses the monodromy around edges: for  $e \in \mathcal{E}_D$  with meridian  $m_e$ , let  $\rho^*(e) = \psi^{-1}(\bar{\rho}(m_e))$ . Now Lemma 3.4 implies

$$\psi(\rho^*(-e)) = \bar{\rho}(m_{-e}) = \bar{\rho}(m_{-e}^+) \bar{\rho}(m_{-e}^-)^{-1} = \bar{\rho}(m_e^+)^{-1} \bar{\rho}(m_e^-) = \psi(\rho^*(e)^{-1}).$$

Moreover, at a vertex  $v$  as shown in Figure 1, the two first lines of Equation (4) imply  $\varphi_+(\rho^*(b'a')) = \varphi_+(\rho^*(ab))$  and  $\varphi_-(\rho^*(b'a')) = \varphi_-(\rho^*(ab))$ . But clearly the map  $\varphi_+ \times \varphi_- : G^* \rightarrow \bar{G} \times \bar{G}$  is injective as  $\psi$  – which is bijective – factors through this map. Therefore, we have  $\rho^*(b'a') = \rho^*(ab)$ . The last line of Equation (4) shows  $\varphi_+(\rho^*(b')) \varphi_-(\rho^*(a')) = \varphi_-(\rho^*(a)) \varphi_+(\rho^*(b))$ . Thus,  $\rho^*$  is a  $G^*$ -coloring of  $D$ .

Furthermore, if  $\rho$  is a  $\mathbf{Y}$ -admissible representation then since  $\mathbf{Y}$  is a union of conjugacy classes in  $\bar{G}$  we have  $\bar{\rho}(m_e) = \bar{\rho}(\vec{R}_i^+)^{-1} \rho(\vec{R}_j \overleftarrow{R}_i) \bar{\rho}(\vec{R}_i^+) \in \bar{\rho}(\vec{R}_i^+)^{-1} \mathbf{Y} \bar{\rho}(\vec{R}_i^+) \subset \mathbf{Y}$ . Since  $\mathbf{Y}^* = \psi^{-1}(\mathbf{Y})$  we have  $\rho^*$  is a  $G^*$ -coloring with values in  $\mathbf{Y}^*$ .

Conversely, suppose that  $\rho^*$  is a  $G^*$ -coloring of  $D$ . The definition of a  $G^*$ -coloring implies that the map  $\bar{\rho} : \bar{\pi} \rightarrow \bar{G}$  defined by  $\bar{\rho}(m_e^\pm) = \varphi_\pm(\rho^*(e))$  satisfy the relations of Equation (4) and thus defines a well defined groupoid morphism. The morphism  $\bar{\rho}$  is  $G$ -valued on meridians and their products. Therefore, if  $R_i$  and  $R_j$  are adjacent regions, then the loop  $\vec{R}_j \overleftarrow{R}_i$  is conjugated to a meridian and since  $G$  is a normal subgroup of  $\bar{G}$  we have  $\bar{\rho}(\vec{R}_j \overleftarrow{R}_i) \in G$ . As these loops generate  $\pi_1(S^3 \setminus L, \infty)$ , we get that the restriction of  $\bar{\rho}$  to  $\pi_1(S^3 \setminus L, \infty)$  takes values in  $G$ . Finally, if  $\rho^*$  is a  $G^*$ -coloring with values in  $\mathbf{Y}^*$ , then the restriction of  $\bar{\rho}$  to  $\pi_1(S^3 \setminus L, \infty)$  takes values in  $\mathbf{Y}$ , since  $\mathbf{Y}$  is a union of conjugacy classes in  $\bar{G}$ . 3.1

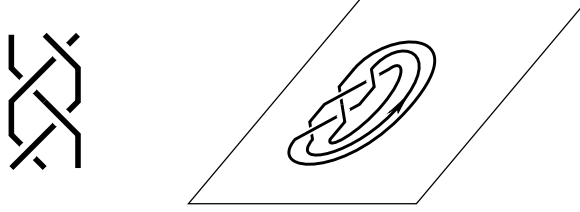


FIGURE 3. Diagrams of the braid word  $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$  and of its braid closure which is a regular projection of the figure eight knot.

*Proof of Proposition 3.2.* Let  $x, y \in G^*$ . Let  $(x', y') \in G^*$  be the unique tuple which is a solution to the system of equations

$$(5) \quad y'x' = xy \text{ and } \varphi_+(y')\varphi_-(x') = \varphi_-(x)\varphi_+(y)$$

(note these equations are exactly part of the requirements of a  $G^*$ -coloring, see Equations (2)). These equations are equivalent to the equation:  $\mathcal{R}(x, y) = (x', y')$ . So the map  $\mathcal{R}$  is determined by Equations (5). Using this, for  $x, y, z \in G^*$ , we have

$$\begin{aligned} (x', y', z') = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}(x, y, z) &\iff \begin{cases} z'y'x' = xyz \\ \varphi_+(z')\varphi_-(y'x') = \varphi_-(xy)\varphi_+(z) \\ \varphi_+(z'y')\varphi_-(x') = \varphi_-(x)\varphi_+(yz) \end{cases} \\ &\iff (x', y', z') = \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}(x, y, z) \end{aligned}$$

Finally, since  $\psi(x_L(x, y))$  is conjugated to  $\psi(x)$  and  $\psi(x_R(x, y))$  is conjugated to  $\psi(y)$  we have that  $\mathcal{R}(Y^* \times Y^*) = Y^* \times Y^*$ . 3.2

*Proof of Proposition 3.3.* Let  $\sigma \in B_n$  be a braid whose braid closure is isotopic to  $L$ . Represent the closure of  $\sigma$  by a planar diagram  $D$  encoded by the braid word corresponding to  $\sigma$  in the generators  $\{\sigma_i\}_{i=1 \dots n-1}$  (see Figure 3). Proposition 3.1 gives a one-to-one correspondence between  $G$ -valued representations on  $L$  and  $G^*$ -colorings of  $D$ . So to prove the proposition it suffices to show that the  $G^*$ -colorings of  $D$  are in one-to-one correspondence with  $G^*$ -coloring of  $\sigma$ .

Let  $y = (y_1, \dots, y_n)$  be a  $G^*$ -coloring of  $\sigma$ , i.e.  $\sigma_{\mathcal{R}}(y) = y$ . Then we color the edges of  $D$  at the bottom of the diagram by  $y_1, \dots, y_n$  and use  $\mathcal{R}$  and its inverse to propagate these colors to the top of the diagram. The result is a  $G^*$ -coloring of  $D$ . Conversely, for any  $G^*$ -coloring of the diagram  $D$ , one obtains a  $G^*$ -coloring of  $\sigma$ . These are clearly inverse correspondences. 3.3

#### 4. $G^*$ -MARKOV TRACES

In this section we give the notion of  $G^*$ -Markov moves and coloring systems. We fix in this section a factorized group  $(G, \overline{G}, G^*, \varphi_+, \varphi_-)$ ,  $Y \subset G$  a union of conjugacy classes in  $\overline{G}$  and  $Y^* = \psi^{-1}(Y)$ . Recall the following well known theorem of Markov.

**Theorem 4.1** (Markov). *Any unframed oriented link in  $S^3$  can be represented as the closure of a braid. Furthermore, any two braids whose closures are ambient isotopic links are related by a finite sequence of the following Markov moves:*

$$\begin{aligned} \sigma\sigma' &\longleftrightarrow \sigma'\sigma \\ \sigma &\longleftrightarrow \sigma_n^{\pm 1}\sigma \end{aligned}$$

where  $n \in \mathbb{N}^*$ ,  $\sigma, \sigma' \in B_n$  and  $\sigma_n$  is the  $n^{\text{th}}$  generator of  $B_{n+1}$ .

The following proposition is a  $G$ -link version of the previous theorem.

**Proposition 4.2.** *Let  $\mathbf{s} : G^* \rightarrow G^*$  be the map defined by  $\mathbf{s}(x) = \psi^{-1}(\varphi_{-}(x)^{-1}\psi(x)\varphi_{-}(x))$ . Any unframed oriented  $G$ -link in  $S^3$  can be represented as the closure of a  $G^*$ -colored braid. Furthermore, any two  $G^*$ -colored braids whose closures are ambient isotopic  $G$ -links are related by a finite sequence of the following  $G^*$ -Markov moves:*

$$\begin{aligned} (\sigma\sigma', \underline{y}) &\longleftrightarrow (\sigma'\sigma, \sigma_{\mathcal{R}}(\underline{y})) \\ (\sigma, \underline{y}) &\longleftrightarrow (\sigma_n^{\pm 1}\sigma, (y_1, \dots, y_n, \mathbf{s}^{\pm 1}(y_n))) \end{aligned}$$

where  $\underline{y} = (y_1, \dots, y_n)$ .

*Proof.* Let  $L$  be a  $G$ -link. Then by Theorem 4.1 the link underlying  $L$  can be represented as the closure of a braid  $\sigma$ . Then Proposition 3.3 implies that the  $G$ -valued representation on  $L$  gives a  $G^*$ -coloring on  $\sigma$ . This proves the first part of the proposition.

To prove the second part of the proposition we will first prove that the  $G^*$ -Markov moves preserve the isotopy class of the closure of the corresponding braids. For the first move, take a diagram  $D$  obtained as the closure of the concatenation of a diagram representing  $\sigma$  and a diagram representing  $\sigma'$ . The representation on  $L$  induces a unique  $G^*$ -coloring on  $D$ , which can be used to deduce unique  $G^*$ -coloring on both  $\sigma\sigma'$  and  $\sigma'\sigma$ . Clearly if the colors of the strands at the bottom of  $\sigma$  are given by  $\underline{y}$  then the colors of the strands at the top of  $\sigma$  (i.e. the bottom of  $\sigma'$ ) are given by  $\sigma_{\mathcal{R}}(\underline{y})$ .

For the second move, take a diagram  $D$  obtained as the closure of a diagram representing  $\sigma$  and consider its  $G^*$ -coloring. Then a diagram  $D'$  for  $\sigma' = \sigma_n\sigma$  is obtained by doing a Reidemeister I move to  $D$ . The  $G^*$ -coloring of  $D'$  is the coloring on  $D$  except with an additional color  $x$  on the loop coming from the Reidemeister move. Now Equation (2) at the new crossing of  $D'$  implies that,

$$\begin{aligned} (x, y_n) = \mathcal{R}(y_n, x) &\iff \varphi_{+}(y_n)\varphi_{-}(x) = \varphi_{-}(y_n)\varphi_{+}(x) \iff \\ \psi(x) = \varphi_{-}(y_n)^{-1}\varphi_{+}(y_n) &\iff x = \mathbf{s}(y_n). \end{aligned}$$

Similarly if  $\sigma' = \sigma_n^{-1}\sigma$ , we have  $(x, y_n) = \mathcal{R}^{-1}(y_n, x) \iff x = \mathbf{s}^{-1}(y_n)$ . Thus, the two  $G^*$ -colorings of the two braids in the second move are related as in the proposition. Moreover, from what we just proved  $\mathbf{s}^{\pm 1}(y_n)$  is the unique value of  $G^*$  which satisfies the second move.

Finally, let  $\sigma$  and  $\sigma'$  be two  $G^*$ -colored braids whose closures are ambient isotopic to the  $G$ -link  $L$ . By Theorem 4.1 the braids  $\sigma$  and  $\sigma'$  are related by a finite sequence of Markov moves. Then from what we just proved the  $G^*$ -coloring is uniquely propagated through these moves by the corresponding  $G^*$ -Markov move. Thus,  $\sigma$  and  $\sigma'$  are related by a finite sequence of  $G^*$ -Markov moves. 4.2

Next we give the notion of a coloring system associated to the pair  $(Y^*, \mathcal{R})$ . Let  $\mathcal{C}$  be a  $\mathbb{k}$ -linear tensor category. Let  $I$  be a set. Let  $\{A_y^i\}_{i \in I, y \in Y^*}$  be a family of objects in  $\mathcal{C}$ . Let  $\mathbf{A} = \{A_y^i : i \in I, y \in Y^*\}$  and  $\mathbf{A}^{\otimes n} = \{V_1 \otimes \dots \otimes V_n : (V_1, \dots, V_n) \in \mathbf{A}^n\}$ .

A family of invertible morphisms

$$C_{y,z}^{i,j} : A_y^i \otimes A_z^j \rightarrow A_{\mathbf{x}_{\mathcal{R}}(y,z)}^j \otimes A_{\mathbf{x}_{\mathcal{L}}(y,z)}^i$$

in  $\mathcal{C}$  indexed by  $y, z \in Y^*$  and  $i, j \in I$  form a *holonomy braiding* if for all  $x, y, z \in Y^*$  and  $i, j, k \in I$  the following equality holds

$$\left(C_{y',z'}^{j,k} \otimes \text{Id}\right) \left(\text{Id} \otimes C_{x',z}^{i,k}\right) (C_{x,y}^{i,j} \otimes \text{Id}) = \left(\text{Id} \otimes C_{x'',y''}^{i,j}\right) \left(C_{x,z''}^{i,k} \otimes \text{Id}\right) (\text{Id} \otimes C_{y,z}^{j,k})$$

where  $x' = \mathbf{x}_{\mathcal{L}}(x, y)$ ,  $y' = \mathbf{x}_{\mathcal{R}}(x, y)$ ,  $z' = \mathbf{x}_{\mathcal{R}}(x', z)$ ,  $x'' = \mathbf{x}_{\mathcal{L}}(x, z'')$ ,  $y'' = \mathbf{x}_{\mathcal{L}}(y, z)$  and  $z'' = \mathbf{x}_{\mathcal{R}}(y, z)$ . A family of linear maps  $\mathbf{t} = \{t_V : \text{End}_{\mathcal{C}}(V) \rightarrow \mathbb{k}\}$  indexed by  $V \in \mathbf{A}^{\otimes n}$  and  $n \in \mathbb{N}^* = \{1, 2, 3, \dots\}$  is a  $G^*$ -Markov trace if for all  $n \in \mathbb{N}^*$  and  $V, W \in \mathbf{A}^{\otimes n}$  the following hold

$$(1) \quad t_W(fg) = t_V(gf) \text{ for any } f : V \rightarrow W, g : W \rightarrow V,$$



- (2) there exists a set of *twists*  $\{\theta_y^i\}_{i \in I, y \in Y^*}$  of invertible elements of  $\mathbb{k}$  such that
- (a) for  $y, y' \in Y^*$ , if  $\psi(y)$  and  $\psi(y')$  are conjugate in  $\overline{G}$  then  $\theta_y^i = \theta_{y'}^i$  for all  $i \in I$ ,
  - (b)  $\mathbf{t}_{V \otimes A_y^i \otimes A_{s(y)}^i}((\text{Id} \otimes (C_{y,s(y)}^{i,i})^{\pm 1})(h \otimes \text{Id})) = (\theta_y^i)^{\pm 1} \mathbf{t}_{V \otimes A_y^i}(h)$  for any  $h \in \text{End}_{\mathcal{C}}(V \otimes A_y^i)$ .

A tuple  $(Y^*, \mathcal{R}, \{A_y^i\}, \{C_{y,z}^{i,j}\}, \mathbf{t})$  is a *trace coloring system* if  $\{C_{y,z}^{i,j}\}$  is a holonomy braiding and  $\mathbf{t}$  is a  $G^*$ -Markov trace. Given such a tuple we can assign an endomorphism to a  $Y^*$ -colored braid representing a  $Y$ -admissible  $G$ -link whose components are colored by elements of  $I$ . Consider the pair  $(\sigma_k, \underline{y})$  where  $\sigma_k$  is a generator of  $B_n$  whose  $l^{\text{th}}$  strand is colored by  $i_l \in I$  and  $\underline{y} = (y_1, \dots, y_n) \in Y^{*n}$ . Then  $(\sigma_k, \underline{y})$  induces the homomorphism

$$(6) \quad \text{Id}_{A_{y_1}^{i_1}} \otimes \dots \otimes \text{Id}_{A_{y_{k-1}}^{i_{k-1}}} \otimes C_{y_k, y_{k+1}}^{i_k, i_{k+1}} \otimes \text{Id}_{A_{y_{k+2}}^{i_{k+2}}} \otimes \dots \otimes \text{Id}_{A_{y_n}^{i_n}}.$$

Let  $B_n^{Y^*, I}$  be the set of  $Y^*$ -braids  $(\sigma, \underline{y})$  with  $n$  strands such that the components of the closure of  $\sigma$  are colored by elements of  $I$ . Let  $(\sigma, \underline{y}) \in B_n^{Y^*, I}$  then the  $I$ -coloring induces a map from the strands of  $\sigma$  to  $I$  (say the  $l^{\text{th}}$  strand is colored by  $i_l \in I$ ). Thus, Equation (6) induces an endomorphism

$$f_{(\sigma, \underline{y})} : A_{y_1}^{i_1} \otimes A_{y_2}^{i_2} \otimes \dots \otimes A_{y_n}^{i_n} \rightarrow A_{y_1}^{i_1} \otimes A_{y_2}^{i_2} \otimes \dots \otimes A_{y_n}^{i_n}$$

where  $\underline{y} = (y_1, \dots, y_n)$ . We write  $\mathbf{t}(f_{(\sigma, \underline{y})}) = \mathbf{t}_{A_{y_1}^{i_1} \otimes \dots \otimes A_{y_n}^{i_n}}(f_{(\sigma, \underline{y})})$ .

**Theorem 4.3.** *Let  $(Y^*, \mathcal{R}, \{A_y^i\}, \{C_{y,z}^{i,j}\}, \mathbf{t})$  be a trace coloring system in a  $\mathbb{k}$ -linear tensor category. Let  $(L, \rho)$  be a  $Y$ -admissible  $G$ -link whose components are colored with elements of  $I$ . Let  $(\sigma, \underline{y})$  be a  $Y^*$ -colored braid representing  $L$ . Then*

$$F'(L, \rho) = \mathbf{t}(f_{(\sigma, \underline{y})})$$

*is independent of the choice of  $(\sigma, \underline{y})$  and yields a well defined invariant of the  $G$ -link  $(L, \rho)$ .*

## 5. MODIFIED RIGHT TRACES ON RIGHT IDEALS

There are many examples where the usual categorical trace is generically zero and invariant of Theorem 4.3 is trivial with this trace. Here we recall the modified trace construction given in [11]. These traces can be non-zero when the usual categorical trace is zero. Later in the paper we will show that the modified traces lead to  $G^*$ -Markov traces and non-trivial link invariants.

In this section we recall some properties about traces on ideals, for more details see [11]. Let  $\mathbb{k}$  be a domain. A *monoidal  $\mathbb{k}$ -category* is a strict monoidal category  $\mathcal{C}$  such that its hom-sets are  $\mathbb{k}$ -modules, the composition and monoidal product of morphisms are  $\mathbb{k}$ -bilinear, and  $\text{End}_{\mathcal{C}}(\mathbb{I})$  is a free  $\mathbb{k}$ -module of rank one, where  $\mathbb{I}$  is the unit object. An object  $X$  of a monoidal  $\mathbb{k}$ -category  $\mathcal{C}$  is *simple* if  $\text{End}_{\mathcal{C}}(X)$  is a free  $\mathbb{k}$ -module of rank 1. Equivalently,  $X$  is simple if the  $\mathbb{k}$ -homomorphism  $\mathbb{k} \rightarrow \text{End}_{\mathcal{C}}(X)$ ,  $k \mapsto k \text{Id}_X$  is an isomorphism.

Let  $\mathcal{C}$  be a pivotal  $\mathbb{k}$ -category (see for example [2, 16]), with unit object  $\mathbb{I}$ , duality morphisms

$$\begin{aligned} \text{ev}_X : X^* \otimes X &\rightarrow \mathbb{I}, & \text{coev}_X : \mathbb{I} &\rightarrow X \otimes X^*, \\ \widetilde{\text{ev}}_X : X \otimes X^* &\rightarrow \mathbb{I}, & \widetilde{\text{coev}}_X : \mathbb{I} &\rightarrow X^* \otimes X. \end{aligned}$$

Recall that in  $\mathcal{C}$ , the left dual and right dual of a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  coincide:

$$\begin{aligned} f^* &= (\text{ev}_Y \otimes \text{Id}_{X^*})(\text{Id}_{Y^*} \otimes f \otimes \text{Id}_{X^*})(\text{Id}_{Y^*} \otimes \text{coev}_X) \\ &= (\text{Id}_{X^*} \otimes \widetilde{\text{ev}}_Y)(\text{Id}_{X^*} \otimes f \otimes \text{Id}_{Y^*})(\widetilde{\text{coev}}_X \otimes \text{Id}_{Y^*}) : Y^* \rightarrow X^*. \end{aligned}$$

For  $X, Y, Z \in \mathcal{C}$ , the *right partial trace* (with respect to  $X$ ) is the map  $\text{tr}_r^X : \text{Hom}_{\mathcal{C}}(Y \otimes X, Z \otimes X) \rightarrow \text{Hom}_{\mathcal{C}}(Y, Z)$  defined, for  $g \in \text{Hom}_{\mathcal{C}}(Y \otimes X, Z \otimes X)$  by

$$\text{tr}_r^X(g) = (\text{Id}_Z \otimes \widetilde{\text{ev}}_X)(g \otimes \text{Id}_{X^*})(\text{Id}_Y \otimes \text{coev}_X).$$

By a *retract* of an object  $X$  of a category  $\mathcal{C}$ , we mean an object  $U$  of  $\mathcal{C}$  such that there exists morphisms  $p: X \rightarrow U$  and  $q: U \rightarrow X$  verifying  $pq = \text{Id}_U$ . By a *right ideal* of a monoidal category  $\mathcal{C}$ , we mean a class  $\mathcal{I} \subset \mathcal{C}$  such that the following two conditions hold:

- (1) If  $X \in \mathcal{I}$  and  $Y \in \mathcal{C}$  then  $X \otimes Y \in \mathcal{I}$ .
- (2) Any retract (in  $\mathcal{C}$ ) of an object of  $\mathcal{I}$  belongs to  $\mathcal{I}$ .

Given an object  $X$  of  $\mathcal{C}$  we can define the right ideal  $\mathcal{I}_X^r$  as follows

$$\mathcal{I}_X^r = \{U \in \mathcal{C} \mid U \text{ is a retract of } X \otimes Z \text{ for some } Z \in \mathcal{C}\}.$$

A *right trace* on a right ideal  $\mathcal{I}$  of  $\mathcal{C}$  is a family  $\mathbf{t} = \{\mathbf{t}_X: \text{End}_{\mathcal{C}}(X) \rightarrow \mathbb{k}\}_{X \in \mathcal{I}}$  of  $\mathbb{k}$ -linear forms such that

$$\mathbf{t}_{X \otimes Z}(f) = \mathbf{t}_X(\text{tr}_r^Z(f)) \quad \text{and} \quad \mathbf{t}_V(gh) = \mathbf{t}_U(hg)$$

for any  $f \in \text{End}_{\mathcal{C}}(X \otimes Z)$ ,  $g \in \text{Hom}_{\mathcal{C}}(U, V)$ , and  $h \in \text{Hom}_{\mathcal{C}}(V, U)$ , with  $X, U, V \in \mathcal{I}$  and  $Z \in \mathcal{C}$ .

Since  $\mathcal{C}$  is pivotal, then  $\phi = \{\phi_X = (\tilde{\text{ev}}_X \otimes \text{Id}_{X^{**}})(\text{Id}_X \otimes \text{coev}_{X^*}): X \rightarrow X^{**}\}_{X \in \mathcal{C}}$  is a monoidal natural isomorphism. Let  $X$  be an object of  $\mathcal{C}$  and  $t: \text{End}_{\mathcal{C}}(X) \rightarrow \mathbb{k}$  a  $\mathbb{k}$ -linear map. We say that  $t$  is a *right ambidextrous trace* on  $X$  if

$$t(\phi_X^{-1}(\text{tr}_r^X(f))^* \phi_X) = t(\text{tr}_l^{X^*}(f))$$

for all  $g \in \text{End}_{\mathcal{C}}(X^* \otimes X)$ .

If  $X$  is a simple object of  $\mathcal{C}$ , we denote by  $\langle \rangle_X: \text{End}_{\mathcal{C}}(X) \rightarrow \mathbb{k}$  the inverse of the  $\mathbb{k}$ -linear isomorphism  $\mathbb{k} \rightarrow \text{End}_{\mathcal{C}}(X)$  defined by  $k \mapsto k \text{Id}_X$ . We say that an object  $X$  of  $\mathcal{C}$  is *right ambi* if it is simple and the  $\mathbb{k}$ -linear form  $\langle \rangle_X$  is a right ambidextrous trace on  $X$ .

**Theorem 5.1** ([11]). *If  $t$  is a right ambidextrous trace on an object  $X$  of  $\mathcal{C}$ , then there exists a unique right trace  $\mathbf{t}$  on  $\mathcal{I}_X^r$  such that  $\mathbf{t}_X = t$ .*

*Proof.* The theorem is a special case of Theorem 10 in [11] where  $\mathcal{O} = \{X\}$ . 5.1

## 6. EXAMPLE : SEMI-CYCLIC MODULES OF $U_{\xi}$

In this section we prove that the so called semi-cyclic modules of  $U_{\xi}$  can be used to define a trace coloring system. We do this in six steps (contained in six subsections): 1) we define several algebras, 2) consider a completion, 3) use this completion to define a  $R$ -matrix, 4) show that this  $R$ -matrix acts on certain modules giving a holonomy braiding, 5) define a right trace and 6) combine the results of the section to define a trace coloring system. 7) is an explicit computation of the holonomy  $R$ -matrix at fourth root of unity.

In this section we consider seven versions of quantized  $\mathfrak{sl}(2)$ :

$$\begin{array}{ccccccc} U_q^D & \subset & \widehat{U_q^D} & \subset & U_h(\mathfrak{sl}(2)) \\ & & \downarrow & & \downarrow \\ U_{\xi} & \subset & U_{\xi}^H & \subset & U_{\xi}^D & \subset & \widehat{U_{\xi}^D} \end{array}$$

The character “ $h$ ” indicates  $U_h(\mathfrak{sl}(2))$  is a  $\mathbb{C}[[h]]$ -topological algebra, the character “ $q$ ” indicates that the corresponding algebra is a  $\mathbb{Z}[q^{\pm 1}]$  algebra and the character “ $\xi$ ” indicates a specialization at a root of unity  $\xi \in \mathbb{C}$ . The character “ $D$ ” is for divided power (of the generator  $F$ ) and the hat suggests a completion.

**6.1. An integral version of  $U_q \mathfrak{sl}(2)$ .** We use the following notation.

$$(7) \quad \{x\}_q = q^x - q^{-x} \quad \{x; n\}_q! = \{x\}_q \{x-1\}_q \cdots \{x-n+1\}_q \quad \{n\}_q! = \{n; n\}_q! \quad \left[ \begin{matrix} x \\ k \end{matrix} \right]_q = \frac{\{x; k\}_q!}{\{k\}_q!}.$$

6.1.1. *Generic  $q$ .* Let  $q = e^{h/2} \in \mathbb{C}[[h]]$  and  $q^x = e^{xh/2}$ . Let  $U_h(\mathfrak{sl}(2))$  be the Drinfeld-Jimbo quantization of  $\mathfrak{sl}(2)$  generated over  $\mathbb{C}[[h]]$  by  $H, E, F$  with relations:

$$(8) \quad [H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

where  $K = q^H$ . Here  $U_h(\mathfrak{sl}(2))$  has the  $h$ -adic topology, i.e. the topology coming from the inverse limit  $\lim_{+\infty \leftarrow n} U_h(\mathfrak{sl}(2))/h^n U_h(\mathfrak{sl}(2))$ . As a vector space  $U_h(\mathfrak{sl}(2))$  is isomorphic to  $U(\mathfrak{sl}(2))[[h]]$ .

The algebra  $U_h(\mathfrak{sl}(2))$  is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K, & \Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \epsilon(E) &= \epsilon(F) = \epsilon(H) = 0, & S(E) &= -EK^{-1}, & S(F) &= -KF. \end{aligned}$$

In addition,  $U_h(\mathfrak{sl}(2))$  is a braided Hopf algebra with  $R$ -matrix

$$R_h = q^{H \otimes H/2} \sum_{n=0}^{\infty} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n,$$

Following the ideas of Habiro<sup>1</sup> in [13] we define  $U_q^D$  as the  $\mathbb{Z}[q, q^{-1}]$ -sub-Hopf algebra of  $U_h \mathfrak{sl}(2)$  generated by  $H, K, K^{-1}, E$  and  $F^{[n]} = \frac{\{1\}^{2n}}{\{n\}!} F^n$ ,  $n \in \mathbb{N}$ . Here the  $D$  stands in  $U_q^D$  for the divided powers of  $F$ .

**Proposition 6.1.** *As a  $\mathbb{Z}[q, q^{-1}]$ -algebra  $U_q^D$  is generated by  $H, K, K^{-1}, E, F^{[n]}$ ,  $n \in \mathbb{N}$  with the relations*

$$(9) \quad [H, E] = 2E, \quad KE = q^2 EK, \quad [K, H] = 0, \quad [H, F^{[n]}] = -2nF^{[n]},$$

$$(10) \quad F^{[m]} F^{[n]} = \begin{bmatrix} m+n \\ n \end{bmatrix}_q F^{[m+n]}, \quad KF^{[n]} = q^{-2n} F^{[n]} K,$$

$$(11) \quad E^m F^{[n]} = \sum_{k=0}^{\min(m,n)} \begin{bmatrix} m \\ k \end{bmatrix}_q F^{[n-k]} \{H - m - n + 2k; k\}_q! E^{m-k}.$$

Moreover,  $U_q^D$  has a PBW type basis over  $\mathbb{Z}[q, q^{-1}]$  given by basis elements of the form  $F^{[a]} K^b H^c E^d$  for  $a, c, d \in \mathbb{N}$  and  $b \in \mathbb{Z}$ .

*Proof.* The proof is essentially the same as the proof of Proposition 3.2 in [12]: We have that relations (9), (10) and (11) hold in  $U_q^D$ . One can prove using only these relations that  $U_q^D$  is  $\mathbb{Z}[q, q^{-1}]$ -spanned by the elements  $F^{[a]} K^b H^c E^d$  for  $a, c, d \in \mathbb{N}$  and  $b \in \mathbb{Z}$ . Thus, the  $\mathbb{Z}[q, q^{-1}]$ -algebra is given by these generators and relations. Moreover, this proof shows the algebra has the PBW basis as described above. 6.1

The Hopf algebra structure of  $U_h(\mathfrak{sl}(2))$  induces a Hopf algebra structure on  $U_q^D$ , in particular:

$$(12) \quad \Delta(F^{[n]}) = \sum_{k=0}^n q^{k(n-k)} F^{[k]} K^{k-n} \otimes F^{[n-k]}, \quad S(F^{[n]}) = (-1)^n q^{-n(n-1)} F^{[n]} K^n.$$

Remark that the  $F^{[n]}$  is small in  $U_h \mathfrak{sl}(2)$  for the  $h$ -adic topology.

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<sup>1</sup>Habiro uses a isomorphic version of  $U_q \mathfrak{sl}(2)$  obtained by sending  $E \mapsto F$ ,  $F \mapsto E$ ,  $K \mapsto K^{-1}$  and  $q \mapsto v$ .

6.1.2. *Specialization of  $q$  to a root of unity.* Fix a positive integer  $N$  and let  $\xi = e^{\frac{2i\pi}{N}}$  be a  $N^{\text{th}}$ -root of unity. Let  $r = N/2$  if  $N$  is even and  $r = N$  if  $N$  is odd. Then  $r$  is the smallest positive integer such that  $\{r\}_\xi = 0$ . If  $x \in \mathbb{C}$  then let  $\xi^x = e^{\frac{2ix\pi}{N}}$ . The number  $\xi^{-\frac{r(r-1)}{2}}$  shows up often in what follows, for this reason we give it a special notation: let  $\sigma = \xi^{-\frac{r(r-1)}{2}}$  which is 1 if  $N$  odd and  $i^{1-r}$  if  $N$  is even.

Let  $U_\xi^D = U_q^D \otimes_{q=\xi} \mathbb{Q}(\xi)$  be the specialization of  $U_q^D$  at the root of unity  $q = \xi$ . The Hopf algebra structure of  $U_q^D$  induces a Hopf algebra structure on  $U_\xi^D$ . Consider the elements  $\mathbf{f} = F^{[r]}$  and  $F = \frac{F^{[1]}}{\{1\}}$  of  $U_\xi^D$ . Then  $F^r = \{r\}_\xi! \mathbf{f} / \{1\}_\xi^{2r} = 0$  in  $U_\xi^D$ . In  $U_\xi^D$  the element  $K^r$  is central but  $E^r$  does not commute with  $H$  and  $\mathbf{f}$ :

$$\begin{aligned} [H, E^r] &= 2rE^r, & [E, \mathbf{f}] &= F^{[r-1]} \{H + 1 - r\}_\xi = \frac{\sigma \xi^r \{1\}_\xi^{2r-2}}{r} F^{r-1} \{H + 1\}_\xi, \\ (13) \quad [E^r, \mathbf{f}] &= \{H; r\}_\xi! = \sigma \{rH\}_\xi. \end{aligned}$$

We will also need two additional algebras. Let  $U_\xi$  be the standard quantization of  $\mathfrak{sl}(2)$ , i.e. the  $\mathbb{C}$ -algebra with generators  $E, F, K, K^{-1}$  and the following defining relations:

$$(14) \quad KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = \xi^2 E, \quad KFK^{-1} = \xi^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{\xi - \xi^{-1}}.$$

Let  $U_\xi^H$  be the  $\mathbb{C}$ -algebra given by the generators  $E, F, K, K^{-1}, H$ , relations (14), and the following additional relations:

$$HK = KH, \quad HK^{-1} = K^{-1}H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

Both  $U_\xi$  and  $U_\xi^H$  are sub-Hopf algebras of  $U_\xi^D$ .

6.1.3. *The category  $\mathcal{D}$  of  $U_\xi$ -modules.* Here we define the category of  $U_\xi$ -modules which will be used to define a trace coloring system later in this section.

Let  $\mathcal{Z}^0$  be the central sub-algebra of  $U_\xi$  generated by the elements  $K^r$  and  $E^r$ . Given a  $U_\xi$ -module  $V$  the weight space corresponding to a *weight*  $\kappa \in \mathbb{C}$  is the set of elements  $v \in V$  such that  $Kv = \kappa v$ . A  $U_\xi$ -module  $V$  is called a *weight module* if  $V$  splits as a direct sum of its weight spaces and if all elements of  $\mathcal{Z}^0$  act diagonally on it. Let  $\mathcal{D}$  be the category of finite dimensional weight modules over  $U_\xi$ .

The category  $\mathcal{D}$  has the following simple modules: let  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$  and  $\varepsilon \in \mathbb{C}$  then the  $r$ -dimensional vector space becomes a simple weight module  $W_{\alpha, \varepsilon}$  whose action is determined by a lowest weight vector  $w$  (i.e.  $Fw = 0$ ) such that

$$Kw = q^{\alpha-r+1}w \text{ and } E^r w = \varepsilon w.$$

Since the action of  $E$  is cyclical but the action of  $F$  is not, we call the modules  $W_{\alpha, \varepsilon}$  *semi-cyclic*. Note here we use the middle weight notation. For a more complete theory of the representation theory of  $U_\xi$ , see [6] and subsequent papers.

There is a  $G^*$ -grading on  $\mathcal{D}$  ( $\mathcal{D}$  fiber over  $G^*$ ) defined by the following : if  $g = \begin{pmatrix} 1 & \varepsilon \\ 0 & \kappa \end{pmatrix} \in G^*$ , then

$$\mathcal{D}_g = \{V \in \mathcal{D} : F^r, E^r, K^r \text{ acts by respectively } 0, \varepsilon \text{Id}_V, \kappa \text{Id}_V\}.$$

For example the degree of  $W_{\alpha, \varepsilon}$  is given by  $\begin{pmatrix} 1 & \varepsilon \\ 0 & \sigma^2 q^{r\alpha} \end{pmatrix}$ .

**6.2. Completion of bigraded bialgebra.** As mentioned above it is the goal of this section to define a trace coloring system using  $\mathcal{D}$  and the semi-cyclic modules. To do this we must define a holonomy braiding for semi-cyclic modules. To this end, in this subsection we consider a completion of graded bi-algebras. In Subsection 6.3 we show that this completion leads to a quasi  $R$ -matrix in the completion of  $(U_\xi^D)^{\otimes 2}$ . This is a general construction because it will be applied to several different algebras, including the tensor product of certain algebras.

Let  $\mathbb{k}$  be a domain. Let  $\mathcal{B} = \bigoplus_{(w,\ell) \in \mathbb{Z} \times \mathbb{N}} \mathcal{B}_{w,\ell}$  be a bigraded  $\mathbb{k}$ -algebra with product  $\cdot$  and unit  $\eta$  such that

$$(15) \quad \eta(\mathbb{k}) \subset \mathcal{B}_{0,0},$$

$$(16) \quad \mathcal{B}_{w,\ell} \cdot \mathcal{B}_{w',\ell'} \subset \bigoplus_{\ell''=\max(\ell,\ell'-w)}^{\ell+\ell'} \mathcal{B}_{w+w',\ell''}$$

for all  $w, w' \in \mathbb{Z}$  and  $\ell, \ell' \in \mathbb{N}$ . If  $\mathcal{B}$  is a bialgebra with coproduct  $\Delta$  and counit  $\epsilon$ , then we assume that the grading is compatible with the coalgebra maps:

$$(17) \quad \epsilon(b) = 0, \text{ for all } b \in \mathcal{B}_{w,\ell} \text{ with } (w, \ell) \neq (0, 0),$$

$$(18) \quad \Delta \mathcal{B}_{w,\ell} \subset \bigoplus_{\substack{w_1 + w_2 = w \\ \ell_1 + \ell_2 = \ell}} \mathcal{B}_{w_1,\ell_1} \otimes \mathcal{B}_{w_2,\ell_2}.$$

For  $w \in \mathbb{Z}$  and  $\ell \in \mathbb{N}$ , consider the subspaces  $\mathcal{F}_{w,\ell}(\mathcal{B}) = \bigoplus_{v \leq w, k \geq \ell} \mathcal{B}_{v,k}$ . Then

$$\mathcal{F}_{w_1,\ell_1}(\mathcal{B}) \cdot \mathcal{F}_{w_2,\ell_2}(\mathcal{B}) \subset \mathcal{F}_{w_1+w_2, \max(\ell_1, \ell_2-w_1)}(\mathcal{B}).$$

Define  $\widehat{\mathcal{B}}$  as the direct limit of the inverse limit of the quotient spaces  $\mathcal{B}/\mathcal{F}_{w,\ell}(\mathcal{B})$ , i.e.

$$\widehat{\mathcal{B}} = \lim_{w \rightarrow +\infty} \lim_{\ell \rightarrow +\infty} \mathcal{B}/\mathcal{F}_{w,\ell}(\mathcal{B}).$$

Thus, for each element  $\widehat{u}$  of  $\widehat{\mathcal{B}}$  there exists a minimal  $w(\widehat{u}) \in \mathbb{Z}$  such that  $\widehat{u}$  is uniquely written as a sum  $\sum_{\ell=0}^{+\infty} u_\ell$  where  $u_\ell \in \bigoplus_{w=-\infty}^{w(\widehat{u})} \mathcal{B}_{w,\ell}$ . Equation (16) implies that the multiplication of  $\mathcal{B}$  extend to  $\widehat{\mathcal{B}}$ . In particular, if  $\widehat{u}', \widehat{u}'' \in \widehat{\mathcal{B}}$  then for each  $\ell \in \mathbb{N}$  only finitely many terms contribute to  $(\widehat{u}' \cdot \widehat{u}'')_\ell$  and all terms of the product  $\widehat{u}' \cdot \widehat{u}''$  have  $\mathbb{Z}$ -degrees smaller than  $w(\widehat{u}') + w(\widehat{u}'')$ .

The assignment  $\mathcal{B} \mapsto \widehat{\mathcal{B}}$  is functorial. In particular, if  $\mathcal{B}'$  is a  $\mathbb{k}$ -bialgebra graded as above then each algebra morphism  $f : \mathcal{B} \rightarrow \mathcal{B}'$  preserving the grading induces an algebra morphism  $\widehat{f} : \widehat{\mathcal{B}} \rightarrow \widehat{\mathcal{B}'}$ . Also, for  $n > 0$ ,  $\mathcal{B}^{\otimes n}$  inherit a  $\mathbb{Z} \times \mathbb{N}$ -grading from  $\mathcal{B}$  (by summing the degrees of the factors). This grading satisfies the relations given in (16). Consider the completion  $\widehat{\mathcal{B}^{\otimes n}}$ . Then, the functoriality implies that the coproduct and counit on  $\mathcal{B}$  naturally induces a coalgebra structure on  $\widehat{\mathcal{B}}$ :

$$\widehat{\Delta} : \widehat{\mathcal{B}} \rightarrow \widehat{\mathcal{B}^{\otimes 2}}, \quad \widehat{\epsilon} : \widehat{\mathcal{B}} \rightarrow \widehat{\mathbb{k}}$$

where  $\widehat{\mathbb{k}} = \mathbb{k}$  has the trivial grading.

**Proposition 6.2.** *For  $w \in \mathbb{Z}$  and  $\ell \in \mathbb{N}$ , let  $(U_q^D)_{w,\ell}$  be the free  $\mathbb{Z}[q, q^{-1}]$ -module with basis*

$$\{F^{[a]} K^b H^c E^d : a, c, d \in \mathbb{N}, b \in \mathbb{Z}, w = 2d - 2a \text{ and } \ell = 2a\}.$$

*Then  $U_q^D = \bigoplus_{(w,\ell) \in \mathbb{Z} \times \mathbb{N}} (U_q^D)_{w,\ell}$  is a bigrading on  $U_q^D$ . Moreover, this bigrading is preserved in  $U_\xi^D$  after specializing  $q = \xi$ . Both of these bigradings satisfy relations (15)–(18) and so there exists bialgebras  $\widehat{U}_q^D$  and  $\widehat{U}_\xi^D$ .*

*Proof.* Equations (15) and (17) are satisfied, since the Hopf algebra structure of  $U_\xi^D$  is induced by  $U_h(\mathfrak{sl}(2))$ . Moreover, it is easy to see from the definition of the coproduct that Equation (18) is satisfied. Finally, Equation (16) follows from using Equation (11) applied to the product of two elements of the PBW base  $\{F^{[a]}K^bH^cE^d\}_{a,c,d \in \mathbb{N}, b \in \mathbb{Z}}$  of  $U_q^D$ . 6.2

To give some feeling for the completions  $\widehat{U_q^D}$  and  $\widehat{U_\xi^D}$  we consider the following example. The subspace  $\mathcal{F}_{-6,2n}(U_q^D)$  is linear combinations of elements of the form  $F^{[a]}K^bH^cE^d$  where  $a \geq n$  and  $d \leq a - 3$ . So  $U_q^D / \mathcal{F}_{-6,2n}(U_q^D)$  contains the element  $c_{n-1}F^{[n-1]}E^{n-4} + c_{n-2}F^{[n-2]}E^{n-5} + \dots + c_3F^{[3]}E^0$  where  $c_i \in \mathbb{Z}[q, q^{-1}]$ . Also, note that  $F^{[n]}E^{n-3} = 0$  in  $U_q^D / \mathcal{F}_{-6,2n}(U_q^D)$ . Thus,  $\lim_{+\infty \leftarrow \ell} U_q^D / \mathcal{F}_{-6,\ell}(U_q^D)$  and  $\widehat{U_q^D}$  contain the element  $\sum_{i=3}^{\infty} c_i F^{[i]}E^{i-3}$ . Also remark that for any  $\widehat{u} \in \widehat{\mathcal{F}_{0,1}(U_\xi^D)}$  and any formal power series  $g(X) \in \mathbb{C}[[X]]$ , the expression  $g(\widehat{u})$  is a convergent series in  $\widehat{U_\xi^D}$ .

**6.3. R-matrix.** In this subsection we show that  $\widehat{U_q^D}$  admits a category of modules which is braided.

**Lemma 6.3.** *The inclusion  $U_q^D \rightarrow U_h(\mathfrak{sl}(2))$  extends uniquely to an injective bialgebra morphism*

$$\widehat{U_q^D} \rightarrow U_h(\mathfrak{sl}(2)).$$

*Proof.* Since  $F^{[\ell]} \in h^\ell U_h(\mathfrak{sl}(2))$  we have that  $\mathcal{F}_{w,2\ell}(U_q^D) \subset h^\ell U_h(\mathfrak{sl}(2))$ . Thus, the series corresponding to  $\widehat{u} \in \widehat{U_q^D}$  converges to a well defined element in the  $h$ -adic topology of  $U_h(\mathfrak{sl}(2))$ . 6.3

Let  $\mathcal{H} : U_h(\mathfrak{sl}(2))^{\otimes 2} \rightarrow U_h(\mathfrak{sl}(2))^{\otimes 2}$  be the automorphism given by conjugation by  $q^{H \otimes H/2}$ . Hence if  $x, y \in U_h(\mathfrak{sl}(2))$  satisfy  $[H, x] = 2mx$  and  $[H, y] = 2ny$  then

$$\mathcal{H}(x \otimes y) = q^{2mn} x K^n \otimes y K^m.$$

Furthermore, with usual notation, we have  $(\text{Id} \otimes \Delta)(\mathcal{H}(x \otimes y)) = \mathcal{H}_{12} \circ \mathcal{H}_{13}(x \otimes (\Delta y))$  and  $(\Delta \otimes \text{Id})(\mathcal{H}(x \otimes y)) = \mathcal{H}_{13} \circ \mathcal{H}_{23}((\Delta x) \otimes y)$ .

Consider the element  $\check{R} = q^{-H \otimes H/2} R_h$  of  $U_h(\mathfrak{sl}(2))^{\otimes 2}$ . The defining properties of the R-matrix  $R_h$  imply the following relations for  $\check{R}$ :

$$(19) \quad \mathcal{H} \circ \text{Ad}_{\check{R}} \circ \Delta = \Delta^{op},$$

$$(20) \quad (\text{Id} \otimes \Delta)\check{R} = \mathcal{H}_{13}^{-1}(\check{R}_{12})\check{R}_{13},$$

$$(21) \quad (\Delta \otimes \text{Id})\check{R} = \mathcal{H}_{23}^{-1}(\check{R}_{13})\check{R}_{23}.$$

The automorphism  $\mathcal{H}$  restrict to an automorphism of the algebra  $(U_q^D)^{\otimes 2}$  such that

$$\mathcal{H}(\mathcal{F}_{w,\ell}((U_q^D)^{\otimes 2})) \subset \mathcal{F}_{w,\ell}((U_q^D)^{\otimes 2})$$

for all  $(w, \ell) \in \mathbb{Z} \times \mathbb{N}$ . Hence  $\mathcal{H}$  extends to an automorphism of the completion  $(\widehat{U_q^D})^{\otimes 2}$ . This automorphism specializes to an automorphism of  $(\widehat{U_\xi^D})^{\otimes 2}$ .

**Theorem 6.4.** *Let  $\check{R}_t = \sum_{n=0}^{r-1} q^{n(n-1)/2} E^n \otimes F^{[n]}$ . Then the elements*

$$\check{R}_q = \sum_{n=0}^{\infty} q^{n(n-1)/2} E^n \otimes F^{[n]} \in (\widehat{U_q^D})^{\otimes 2}$$

$$\check{R}_\xi = \check{R}_t \exp(\sigma^{-1} E^r \otimes \mathbf{f}) = \exp(\sigma^{-1} E^r \otimes \mathbf{f}) \check{R}_t \in (\widehat{U_\xi^D})^{\otimes 2}$$

satisfy Relations (19), (20) and (21) where  $\mathbf{f} = F^{[r]} \in \widehat{U_\xi^D}$ .

*Proof.* The image of the element  $\check{R}_q$  in  $U_h(\mathfrak{sl}(2)) \widehat{\otimes} U_h(\mathfrak{sl}(2))$  is equal to  $\check{R}$ . Thus, it follows that  $\check{R}_q$  satisfies Relations (19)–(21) since  $\check{R}$  does. Moreover, these relations can be specialized at  $q = \xi$ . In particular, the specialization has the following properties:

$$\{x + \ell r\}_q \mapsto \xi^{\ell r} \{x\}_\xi, \quad \frac{\{r\ell\}_q}{\{r\}_q} \mapsto \ell \xi^{r\ell}, \quad F^{[\ell r+k]} \mapsto \frac{\xi^{\frac{\ell(\ell-1)r^2}{2} + \ell r k} \{1\}_\xi^{2k}}{\{k\}_\xi! \ell!} \mathbf{f}^\ell F^k \text{ for } \ell, k \in \mathbb{N}, k < r$$

Finally, the image of  $\check{R}_q$  by the morphism of bialgebra  $(\widehat{U_q^D})^{\otimes 2} \rightarrow (\widehat{U_\xi^D})^{\otimes 2}$  is given by

$$\check{R}_\xi = \sum_{\ell=0}^{\infty} \sum_{n=\ell r+k, k=0}^{r-1} \xi^{\frac{k(k-1)+\ell r(\ell r-1)}{2} + \ell r k} \xi^{\frac{\ell(\ell-1)r^2}{2} + \ell r k} \frac{\{1\}_\xi^{2k}}{\{k\}_\xi! \ell!} E^k \otimes F^k (E^r \otimes \mathbf{f})^\ell,$$

and so

$$\check{R}_\xi = \check{R}_t \sum_{\ell=0}^{\infty} \xi^{\frac{\ell r(\ell r-1)+r^2 \ell(\ell-1)}{2}} \frac{1}{\ell!} (E^r \otimes \mathbf{f})^\ell = \check{R}_t \exp(\sigma^{-1} E^r \otimes \mathbf{f})$$

since  $\xi^{r^2 \ell^2} = \xi^{r^2 \ell}$  implies  $\xi^{\frac{\ell r(\ell r-1)+r^2 \ell(\ell-1)}{2}} = \xi^{\ell \frac{r(r-1)}{2}} = \sigma^{-\ell}$ .

6.4

Recall that in Subsection 6.1.2 we considered the element  $\mathbf{f} = F^{[r]} \in \widehat{U_\xi^D}$ . A  $\widehat{U_\xi^D}$ -module  $\widehat{V}$  is  $\mathbf{f}$ -profinite if  $\widehat{V}/\mathbf{f}^\ell \widehat{V}$  is finite dimensional for all  $\ell \in \mathbb{N}$  and

$$\widehat{V} \simeq \lim_{+\infty \leftarrow \ell} \widehat{V}/\mathbf{f}^\ell \widehat{V}.$$

Let  $\widehat{\mathcal{C}}$  be the category of  $\mathbf{f}$ -profinite  $\widehat{U_\xi^D}$ -modules. Then  $\widehat{\mathcal{C}}$  is a monoidal category where the tensor product is given by

$$\widehat{V} \widehat{\otimes} \widehat{W} = \lim_{+\infty \leftarrow \ell} (\widehat{V}/\mathbf{f}^\ell \widehat{V}) \otimes (\widehat{W}/\mathbf{f}^\ell \widehat{W}).$$

**Theorem 6.5.** *The category  $\widehat{\mathcal{C}}$  is braided with braiding  $c_{\widehat{V}, \widehat{W}} : \widehat{V} \widehat{\otimes} \widehat{W} \rightarrow \widehat{W} \widehat{\otimes} \widehat{V}$  given by*

$$c_{\widehat{V}, \widehat{W}}(v \otimes w) = \tau(R_\xi(v \otimes w))$$

where  $R_\xi = \xi^{H \otimes H/2} \check{R}_\xi$  and  $\tau : \widehat{V} \widehat{\otimes} \widehat{W} \rightarrow \widehat{W} \widehat{\otimes} \widehat{V}$  is the linear flip map that exchange the two factors.

*Proof.* Since  $H \mathbf{f}^\ell = \mathbf{f}^\ell H - 2r\ell \mathbf{f}^\ell$  we have the action of  $H$  stabilize  $\mathbf{f}^\ell \widehat{V}$ . Thus  $H$  induces a well defined bounded operator on  $\widehat{V}/\mathbf{f}^\ell \widehat{V}$ . Therefore,  $\frac{i\pi}{N} H \otimes H$  is a bounded linear map on the finite dimensional  $\mathbb{C}$ -vector space  $(\widehat{V}/\mathbf{f}^\ell \widehat{V}) \otimes (\widehat{W}/\mathbf{f}^\ell \widehat{W})$  and its exponential  $\xi^{H \otimes H/2}$  is a convergent series. These operators commute with the natural maps  $(\widehat{V}/\mathbf{f}^{\ell+1} \widehat{V}) \otimes (\widehat{W}/\mathbf{f}^{\ell+1} \widehat{W}) \rightarrow (\widehat{V}/\mathbf{f}^\ell \widehat{V}) \otimes (\widehat{W}/\mathbf{f}^\ell \widehat{W})$  and this implies that the action of  $\xi^{H \otimes H/2}$  is well defined on  $\widehat{V} \widehat{\otimes} \widehat{W}$ .

Let  $\widehat{V}$  and  $\widehat{W}$  be objects in  $\widehat{\mathcal{C}}$ . Let  $\rho : \widehat{U_\xi^D} \widehat{\otimes} \widehat{U_\xi^D} \rightarrow \text{End}_{\widehat{\mathcal{C}}}(\widehat{V} \widehat{\otimes} \widehat{W})$  be the representation map corresponding to the module structure of  $\widehat{V} \widehat{\otimes} \widehat{W}$ . Then for any  $x \in \widehat{U_\xi^D} \widehat{\otimes} \widehat{U_\xi^D}$ , we have  $\xi^{H \otimes H/2} \rho(x) \xi^{-H \otimes H/2} = \rho(\mathcal{H}(x))$ . Combining this with Equations (19), (20) and (21) for  $\check{R}_\xi$  implies that  $R_\xi$  is an R-matrix operator for the modules in  $\widehat{\mathcal{C}}$ . Thus,  $\widehat{\mathcal{C}}$  is braided where the braiding given by the action of  $\check{R}_\xi$  composed with the flip map.

6.5

**6.4. Holonomy braiding.** In this subsection we will show that a semi-cyclic module can be realized as a submodule of a certain  $\mathbf{f}$ -profinite  $\widehat{U_\xi^D}$ -module. Then we show that the braiding in  $\widehat{\mathcal{C}}$  induces the desired holonomy braiding on such submodules.

Let  $\mathbb{F} = \mathbb{C}[[\mathbf{f}]]$  be the  $\widehat{U_\xi^D}$ -module whose action is given by

$$E = F = 0, \quad K = \text{Id}, \quad \mathbf{f} = \mathbf{f} \cdot \text{Id}, \quad H = -2r\mathbf{f} \frac{d}{d\mathbf{f}}.$$

By a *nilpotent*  $U_\xi^H$ -module  $V$ , we mean a  $U_\xi^H$ -module such that  $E^r$  and  $F^r$  both act by 0 on  $V$ ,  $q^H$  acts as  $K$  on  $V$  and  $V$  splits as a direct sum of its  $H$ -weight space. Let  $\mathcal{C}^H$  be the category of nilpotent  $U_\xi^H$ -modules (for more details see [10]).

Recall that  $U_\xi \subset U_\xi^H \subset U_\xi^D \subset \widehat{U_\xi^D}$ . If  $V$  is a nilpotent  $U_\xi^H$ -module, let  $\mathbb{F}V = \widehat{U_\xi^D} \otimes_{U_\xi^H} V$  be the infinite dimensional induced  $\widehat{U_\xi^D}$ -module. Proposition 6.1 (i.e. the PBW theorem) implies that  $\mathbb{F}V = V[[\mathbf{f}]]$  and  $\mathbb{F}V/\mathbf{f}^\ell \mathbb{F}V \simeq \bigoplus_{n=0}^{\ell-1} \mathbf{f}^n V$ . Therefore,  $\mathbb{F}V$  is a module of  $\widehat{\mathcal{C}}$ . Also, if  $\mathbb{C}$  is the trivial nilpotent  $U_\xi^H$ -module then  $\mathbb{F} = \mathbb{F}\mathbb{C}$ . Thus, the induction  $\widehat{U_\xi^D} \otimes_{U_\xi^H} \cdot$  gives a functor  $F_\mathbb{F} : \mathcal{C}^H \rightarrow \widehat{\mathcal{C}}$ . However, this functor is not monoidal because  $\mathbb{F}(V \otimes W) \neq \mathbb{F}V \widehat{\otimes} \mathbb{F}W$  and  $F_\mathbb{F}(\mathbb{I}) = \mathbb{F}$ .

If  $V, W$  are nilpotent  $U_\xi^H$ -modules then  $V \otimes W$  is a sub- $U_\xi^H$ -module of the  $\widehat{U_\xi^D}$ -module  $\mathbb{F}V \widehat{\otimes} \mathbb{F}W$ . This space is stable by the multiplication by  $R_\xi$  because  $E^r = 0$  on  $W$  and on this space  $R_\xi$  acts as  $\xi^{H \otimes H/2} \check{R}_t$  which is the truncated  $R$ -matrix used to define a braiding on nilpotent  $U_\xi^H$ -modules (see [10]).

**Theorem 6.6.** *Let  $V$  be a nilpotent  $U_\xi^H$ -module. Let  $W_{\alpha, \varepsilon}$  be the semi-cyclic  $U_\xi$ -module corresponding to  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$  and  $\varepsilon \in \mathbb{C}$  (see Subsection 6.1.3). If  $t \in \mathbb{C}$  then the subspace*

$$V^t = \exp(-t\mathbf{f})V \subset \mathbb{F}V$$

*is a finite dimensional  $U_\xi$ -sub-module of  $\mathbb{F}V$ . In addition, if  $t$  is the complex number determined by  $t = \frac{-\sigma\varepsilon}{\{r\alpha\}}$  then the  $U_\xi$ -modules  $(V_\alpha)^t$  and  $W_{\alpha, \varepsilon}$  are isomorphic, where  $V_\alpha$  is the simple nilpotent  $U_\xi^H$ -module with a lowest weight vector  $v_0$  satisfying  $Fv_0 = 0$  and  $Kv_0 = q^{\alpha-r+1}v_0$ .*

*Proof.* First, the subspace  $V^t$  is finite dimensional with dimension  $\dim(V)$ . Also,  $V^t$  is a  $U_\xi$ -module since Equations (13) imply  $U_\xi \exp(-t\mathbf{f}) = \exp(-t\mathbf{f})U_\xi \subset \widehat{U_\xi^D}$ . For example,  $E^r \exp(-t\mathbf{f}) = \exp(-t\mathbf{f})(E^r - \sigma t\{rH\})$ . Moreover, the last equation determines the action of  $E^r$  on  $V^t$ :

$$E^r \exp(-t\mathbf{f})v = \exp(-t\mathbf{f})(E^r - \sigma t\{rH\})v = -\exp(-t\mathbf{f})\sigma t\{rH\}v$$

for any  $v \in V^t$ . So if  $V^t = (V_\alpha)^t$  with  $t = \frac{-\sigma\varepsilon}{\{r\alpha\}}$  then  $E^r v = \varepsilon v$  for all  $v \in (V_\alpha)^t$ . Let  $v_0$  be the lowest weight vector of  $V_\alpha$  as in the statement of the theorem. Then  $\exp(-t\mathbf{f})v_0$  is a lowest weight vector in  $(V_\alpha)^t$  since  $F \exp(-t\mathbf{f})v_0 = \exp(-t\mathbf{f})Fv_0 = 0$  and  $K \exp(-t\mathbf{f})v_0 = \exp(-t\mathbf{f})Kv_0 = q^{\alpha-r+1} \exp(-t\mathbf{f})v_0$ . Thus,  $\exp(-t\mathbf{f})v_0$  generates a  $r$ -dimensional module which is equal to  $(V_\alpha)^t$  and isomorphic to  $W_{\alpha, \varepsilon}$ . 6.6

One should note that if  $t \neq 0$  then  $V^t$  is not a  $U_\xi^H$ -module since  $HV^t \not\subset V^t$  for  $t \neq 0$ . This fact is an important aspect of the existence of the non-trivial holonomy  $R$ -matrix. In particular, the following lemma shows that the map  $\xi^{H \otimes H/2}$  “moves” the tensor product of two of the  $U_\xi$ -sub-modules of Theorem 6.6.

**Lemma 6.7.** *Let  $V_1$  and  $V_2$  be nilpotent  $U_\xi^H$ -modules on which  $K^r$  acts by  $\kappa_1, \kappa_2$  respectively. Then on  $V_1 \otimes V_2 \subset \mathbb{F}V_1 \otimes \mathbb{F}V_2$ , we have*

$$\xi^{H \otimes H/2}(\exp(-t_1\mathbf{f}) \otimes \exp(-t_2\mathbf{f})) = (\exp(-t_1\kappa_2^{-1}\mathbf{f}) \otimes \exp(-t_2\kappa_1^{-1}\mathbf{f}))\xi^{H \otimes H/2}.$$



In particular, the  $U_\xi$ -module  $V_1^{t_1} \otimes V_2^{t_2}$  is sent into the  $U_\xi$ -sub-module  $V_1^{\kappa_2^{-1}t_1} \otimes V_2^{\kappa_1^{-1}t_2} \subset \mathbb{F}V_1 \otimes \mathbb{F}V_2$ . Similarly, on  $V_1 \otimes V_2$ , we have

$$\begin{aligned} \exp(\sigma^{-1}E^r \otimes \mathfrak{f})(\exp(-t_1\mathfrak{f}) \otimes \exp(-t_2\mathfrak{f})) &= (\exp(-t_1\mathfrak{f}) \otimes 1)(\exp(-(t_1\{rH\} + t_2) \otimes \mathfrak{f})) \\ &= \exp(-t_1\mathfrak{f}) \otimes \exp(-((\kappa_1 - 1/\kappa_1)t_1 + t_2)\mathfrak{f}). \end{aligned}$$

*Proof.* We have

$$\xi^{H \otimes H/2}(\mathfrak{f}^n \otimes 1) = (\mathfrak{f}^n \otimes 1)\xi^{(H-2rn) \otimes H/2} = (\mathfrak{f}^n \otimes 1)(1 \otimes K^{-r})^n \xi^{H \otimes H/2} = (\mathfrak{f} \otimes K^{-r})^n \xi^{H \otimes H/2}.$$

Similarly,  $\xi^{H \otimes H/2}(1 \otimes \mathfrak{f}^n) = (K^{-r} \otimes \mathfrak{f})^n \xi^{H \otimes H/2}$ . Thus, these equalities imply the first relation of the lemma. A similar argument implies the second relation of the lemma.  $\square$  6.7

**Theorem 6.8.** *The restriction of  $R_\xi$  on  $V_1^{t_1} \otimes V_2^{t_2}$  is a map*

$$V_1^{t_1} \otimes V_2^{t_2} \rightarrow V_1^{t'_1} \otimes V_2^{t'_2}$$

where  $t'_1 = \kappa_2^{-1}t_1$  and  $t'_2 = (1 - \kappa_1^{-2})t_1 + \kappa_1^{-1}t_2$ . In particular, if  $a_1 = \exp(-t_1\mathfrak{f})v_1 \in V_1^{t_1}$  and  $a_2 = \exp(-t_2\mathfrak{f})v_2 \in V_2^{t_2}$  then

$$(22) \quad R_\xi(a_1 \otimes a_2) = (\exp(-t'_1\mathfrak{f}) \otimes \exp(-t'_2\mathfrak{f}))\xi^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}_\xi!} \xi^{n(n-1)/2} \text{Ad}_{t_1}(E)^n v_1 \otimes F^n v_2$$

where  $\text{Ad}_t(x) = \exp(t\mathfrak{f})x \exp(-t\mathfrak{f})$ .

*Proof.* Recall  $R_\xi = \xi^{H \otimes H/2} \check{R}_\xi$  where  $\check{R}_\xi = \check{R}_t \exp(\sigma^{-1}E^r \otimes \mathfrak{f})$  by Theorem 6.4. Then the theorem follows from the following computation for maps restricted on  $V_1 \otimes V_2$

$$\begin{aligned} R_\xi(\exp(-t_1\mathfrak{f}) \otimes \exp(-t_2\mathfrak{f})) &= \xi^{H \otimes H/2} \check{R}_t \exp(\sigma^{-1}E^r \otimes \mathfrak{f})(\exp(-t_1\mathfrak{f}) \otimes \exp(-t_2\mathfrak{f})) \\ &= \xi^{H \otimes H/2} \check{R}_t(\exp(-t_1\mathfrak{f}) \otimes \exp(-((\kappa_1 - 1/\kappa_1)t_1 + t_2)\mathfrak{f})) \\ &= \xi^{H \otimes H/2}(\exp(-t_1\mathfrak{f}) \otimes \exp(-((\kappa_1 - 1/\kappa_1)t_1 + t_2)\mathfrak{f}))(\text{Ad}_{t_1} \otimes \text{Id})(\check{R}_t) \\ &= (\exp(-t'_1\mathfrak{f}) \otimes \exp(-t'_2\mathfrak{f}))\xi^{H \otimes H/2}(\text{Ad}_{t_1} \otimes \text{Id})(\check{R}_t) \end{aligned}$$

where the second and fourth equalities follow from Lemma 6.7.  $\square$  6.8

Let  $V_1, V_2$  be nilpotent  $U_\xi^H$ -modules. Let  $t_1, t_2 \in \mathbb{C}$  and set  $t'_1 = \kappa_2^{-1}t_1$  and  $t'_2 = \kappa_1^{-1}t_2 + (1 - \kappa_1^{-2})t_1$  as above. Recall the braiding  $c_{\widehat{V}, \widehat{W}}$  of  $\widehat{\mathcal{C}}$  given in Theorem 6.5. Theorem 6.8 implies that the restriction of this braiding to  $V_1^{t_1} \otimes V_2^{t_2} \subset \mathbb{F}V_1 \otimes \mathbb{F}V_2$  gives a  $U_\xi$ -module map

$$(23) \quad C_{V_1^{t_1}, V_2^{t_2}} : V_1^{t_1} \otimes V_2^{t_2} \rightarrow V_2^{t'_2} \otimes V_1^{t'_1}$$

such that the set of such maps satisfy the braid relations.

Furthermore, if  $x, y, x', y'$  are the degree in  $G^*$  of respectively  $V_1^{t_1}, V_2^{t_2}, V_1^{t'_1}, V_2^{t'_2}$  (see Theorem 6.6), one easily gets that  $(x', y') = \mathcal{R}(x, y)$  where  $\mathcal{R}$  is the map of section 3.4.

**6.5. The pivotal structure and the right trace on the ideal of projectives.** Recall the definition of the category  $\mathcal{D}$  given in Subsection 6.1.3.

The category  $\mathcal{D}$  is a pivotal  $\mathbb{C}$ -category where for any object  $V$  in  $\mathcal{D}$ , the dual object and the duality morphisms are defined as follows:  $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  and

$$(24) \quad \begin{aligned} \text{coev}_V : \mathbb{C} &\rightarrow V \otimes V^* \text{ is given by } 1 \mapsto \sum v_j \otimes v_j^*, \\ \text{ev}_V : V^* \otimes V &\rightarrow \mathbb{C} \text{ is given by } f \otimes w \mapsto f(w), \\ \tilde{\text{ev}}_V : V \otimes V^* &\rightarrow \mathbb{C} \text{ is given by } v \otimes f \mapsto f(K^{1-r}v), \\ \widetilde{\text{coev}}_V : \mathbb{C} &\rightarrow V^* \otimes V \text{ is given by } 1 \mapsto \sum v_j^* \otimes K^{r-1}v_j, \end{aligned}$$

where  $\{v_j\}$  is a basis of  $V$  and  $\{v_j^*\}$  is the dual basis of  $V^*$ . Let  $\text{Proj}$  be the full subcategory of  $\mathcal{D}$  consisting of projective  $U_\xi$ -modules. Then  $\text{Proj}$  is an ideal.

**Proposition 6.9.** *Let  $V_0$  be the unique projective irreducible weight module on which  $K^r - 1$  and  $E^r$  vanish. Then  $V_0$  is right ambi.*

*Proof.* Let  $\mathcal{D}_0$  be the full subcategory of  $\mathcal{D}$  consisting of modules on which  $K^r - 1$  and  $E^r$  vanish. Then  $\mathcal{D}_0$  is the category of finite dimensional modules of the “small quantum group”  $u_\xi$  which is the finite dimensional quotient of  $U_\xi$  by the ideal generated by  $K^r - 1$  and  $E^r$ . It is well known that  $u_\xi$  is unimodular so by [9, Lemma 4.2.1 and Corollary 3.2.1], its simple module  $V_0$  is right ambi. 6.9

Let  $\mathfrak{t}$  be the corresponding right trace on  $\text{Proj} = \mathcal{I}_{V_0}$  (see Theorem 5.1). The following proposition give an explicit formula for the trace  $\mathfrak{t}$ .

**Proposition 6.10.** *For any  $f \in \text{End}_{\mathcal{D}}(V_\alpha^t \otimes W)$ , we have*

$$\mathfrak{t}(f) = \frac{\{\alpha\}}{\{r\alpha\}} \text{tr}_{\mathbb{C}}((\text{Id}_{V_\alpha^t} \otimes K^{1-r}) \circ f).$$

*In particular, the modified dimension of  $V_\alpha^t$  is the scalar  $\mathfrak{t}(\text{Id}_{V_\alpha^t}) = \frac{r\{\alpha\}}{\{r\alpha\}}$ .*

*Proof.* Let  $f \in \text{End}_{\mathcal{D}}(V_\alpha^t \otimes W)$ . Since  $V_\alpha^t$  is simple there exists a scalar  $\lambda$  such that  $\text{tr}_r^W(f) = \lambda \text{Id}_{V_\alpha^t}$ . Then the properties of the trace imply that

$$(25) \quad \mathfrak{t}(f) = \mathfrak{t}(\lambda \text{Id}_{V_\alpha^t}) = \lambda \mathfrak{t}(\text{Id}_{V_\alpha^t}).$$

On the other hand, by the definition of  $\text{tr}_r^W$  we have  $\text{tr}_{\mathbb{C}}((\text{Id}_{V_\alpha^t} \otimes K^{1-r}) \circ f) = \text{tr}_{\mathbb{C}}(\lambda \text{Id}_{V_\alpha^t}) = r\lambda$  (the appearance of  $K^{1-r}$  comes from the morphism  $\tilde{\text{ev}}_W$  used in the definition of  $\text{tr}_r^W$ ). Thus, Equation 25 implies that the proposition follows from the computation of  $\mathfrak{t}(\text{Id}_{V_\alpha^t})$ . The braiding (23) gives maps

$$C_{V_\alpha^t, V_0} : V_\alpha^t \otimes V_0 \rightarrow V_0 \otimes V_\alpha^t \text{ and } C_{V_0, V_\alpha^t} : V_0 \otimes V_\alpha^t \rightarrow V_\alpha^t \otimes V_0.$$

The right partial trace of  $C_{V_\alpha^t, V_0} \circ C_{V_0, V_\alpha^t}$  and  $C_{V_0, V_\alpha^t} \circ C_{V_\alpha^t, V_0}$  are scalar endomorphisms. A computation similar to the proof of Theorem 6.11 on highest weight vectors leads to :

$$\text{tr}_r^{V_\alpha^t}(C_{V_\alpha^t, V_0} \circ C_{V_0, V_\alpha^t}) = r \text{Id}_{V_0} \text{ and } \text{tr}_r^{V_0}(C_{V_0, V_\alpha^t} \circ C_{V_\alpha^t, V_0}) = \frac{\{r\alpha\}}{\{\alpha\}} \text{Id}_{V_\alpha^t}.$$

As  $\mathfrak{t}$  has the same value on these two morphisms and by definition  $\mathfrak{t}(\text{Id}_{V_0}) = 1$ , we get the above formula for  $\mathfrak{t}(\text{Id}_{V_\alpha^t})$ . 6.10

**6.6. The trace coloring system.** In this subsection we will show that  $\mathcal{D}$  has a trace coloring system. Let

$$I = \{\lambda : \mathbb{C} \setminus \{-1, 0, 1\} \rightarrow \mathbb{C} \text{ such that } \exp \circ \lambda = \text{Id}_{\mathbb{C} \setminus \{-1, 0, 1\}}\}$$

be a set of inverse functions of the exponential. Let  $(G, \overline{G}, G^*, \varphi_+, \varphi_-)$ ,  $\mathbf{Y}$ ,  $\mathbf{Y}^*$  and  $\mathcal{R}$  be as in Section 3.4. We identify  $\mathbf{Y}^*$  with the set

$$\mathbf{Y}^* \simeq \{(\kappa, \varepsilon) \in (\mathbb{C} \setminus \{-1, 0, 1\}) \times \mathbb{C}\}.$$

For  $\lambda \in I$  and  $(\kappa, \varepsilon) \in \mathbf{Y}^*$  define  $A_{(\kappa, \varepsilon)}^\lambda$  as the  $U_\xi$ -module  $V_\alpha^t$  given in Theorem 6.6 where  $\alpha = \frac{N\lambda(\sigma^2\kappa)}{2i\pi r}$  and  $t = \frac{-\varepsilon}{\sigma(\kappa - \kappa^{-1})}$ . Then the map in Equation (23) defines a holonomy braiding  $\{C_{y,z}^{\lambda, \lambda'}\}_{y,z \in \mathbf{Y}^*, \lambda, \lambda' \in I}$  on the modules  $A_{(\kappa, \varepsilon)}^\lambda$ . Finally, let  $\mathbf{t}$  be the right trace given in Subsection 6.5.

**Theorem 6.11.** *The tuple  $(\mathbf{Y}^*, \mathcal{R}, \{A_y^\lambda\}, \{C_{y,z}^{\lambda, \lambda'}\}, \mathbf{t})$  is a trace coloring system in  $\mathcal{D}$  where the twist element for  $\lambda \in I, y = (\kappa, \varepsilon) \in \mathbf{Y}^*$  is given by*

$$\theta_{(\kappa, \varepsilon)}^\lambda = q^{\frac{\alpha^2 - (r-1)^2}{2}}$$

$$\text{where } \alpha = \frac{N\lambda(\sigma^2\kappa)}{2i\pi r}.$$

*Proof.* First, for  $(\kappa, \varepsilon) \in \mathbf{Y}^*$ , the map  $\mathbf{s}$  of Proposition 4.2 is given by  $\mathbf{s}((\kappa, \varepsilon)) = (\kappa, \varepsilon\kappa^{-1}) \in \mathbf{Y}^*$ .

Next, we need to show that we have a  $G^*$ -Markov trace as defined in Section 4. In particular, we must define the twist element  $\theta_y^\lambda$  for  $\lambda \in I, y \in \mathbf{Y}^*$ . To do this we consider the endomorphism

$$(26) \quad f = \text{tr}_r^{A_{\mathbf{s}(y)}^\lambda} (C_{y, \mathbf{s}(y)}^{\lambda, \lambda}) \in \text{End}(A_y^\lambda).$$

Since  $A_y^\lambda$  is simple there exists a constant  $\theta_y^\lambda$  such that  $f(a) = \theta_y^\lambda a$  for all  $a \in A_y^\lambda$ . We will compute  $\theta_y^\lambda$  directly and show that it satisfies the properties of the twist.

Let  $\lambda \in I$  and  $y = (\kappa, \varepsilon) \in \mathbf{Y}^*$ . As above set  $\alpha = \frac{N\lambda(\sigma^2\kappa)}{2i\pi r}$  and  $t = \frac{-\varepsilon}{\sigma(\kappa - \kappa^{-1})}$ . Let  $V_\alpha$  be the nilpotent  $U_\xi^H$ -module as in Theorem 6.6. Let  $v_0$  be a highest weight vector of  $V_\alpha$ , i.e.  $Ev_0 = 0$  and  $Hv_0 = (\alpha + r - 1)v_0$ . Let  $\{v_i\}$  be a basis of  $V_\alpha$  with  $v_i = v_0$  for  $i = 0$ . Then  $\{a_i = \exp(-\kappa^{-1}tf)v_i\}$  is a basis of  $(V_\alpha)^{\kappa^{-1}t}$ .

We have

$$\begin{aligned} f(\exp(-tf)v_0) &= \text{tr}_r^{A_{\mathbf{s}(y)}^\lambda} \left( C_{y, \mathbf{s}(y)}^{\lambda, \lambda} \right) (\exp(-tf)v_0) \\ &= \text{tr}_r^{V_\alpha^{\kappa^{-1}t}} \left( C_{V_\alpha^t, V_\alpha^{\kappa^{-1}t}} \right) (\exp(-tf)v_0) \\ &= \left( \text{Id}_{V_\alpha^t} \otimes \tilde{\text{ev}}_{V_\alpha^{\kappa^{-1}t}} \right) \left( C_{V_\alpha^t, V_\alpha^{\kappa^{-1}t}} \otimes \text{Id}_{(V_\alpha^{\kappa^{-1}t})^*} \right) \left( \text{Id}_{V_\alpha^t} \otimes \text{coev}_{V_\alpha^{\kappa^{-1}t}} \right) (\exp(-tf)v_0) \\ &= \left( \text{Id}_{V_\alpha^t} \otimes \tilde{\text{ev}}_{V_\alpha^{\kappa^{-1}t}} \right) \left( C_{V_\alpha^t, V_\alpha^{\kappa^{-1}t}} \otimes \text{Id}_{(V_\alpha^{\kappa^{-1}t})^*} \right) \left( \exp(-tf)v_0 \otimes \sum_i a_i \otimes a_i^* \right) \\ &= \left( \text{Id}_{V_\alpha^t} \otimes \tilde{\text{ev}}_{V_\alpha^{\kappa^{-1}t}} \right) \left( \tau R_\xi \otimes \text{Id}_{(V_\alpha^{\kappa^{-1}t})^*} \right) \left( \exp(-tf)v_0 \otimes \sum_i \exp(-\kappa^{-1}tf)v_i \otimes a_i^* \right) \\ &= \left( \text{Id}_{V_\alpha^t} \otimes \tilde{\text{ev}}_{V_\alpha^{\kappa^{-1}t}} \right) \left( \tau \left( \exp(-\kappa^{-1}tf) \otimes \exp(-tf) \xi^{H \otimes H/2} v_0 \otimes v_0 \right) \otimes a_0^* \right) \\ &= q^{(\alpha+r-1)^2/2 + (1-r)(\alpha+r-1)} \exp(-tf)v_0 \end{aligned}$$

where the first five equalities are definitions, the sixth equality comes from Equation (22) combined with the fact that  $Ev_0 = 0$  and the final equality comes from:

$$\tilde{e}v_{V_\alpha^{\kappa^{-1}t}}(\exp(-\kappa^{-1}tf)v_0 \otimes a_0^*) = a_0^*((\exp(-\kappa^{-1}tf)K^{1-r}v_0)) = q^{(1-r)(\alpha+r-1)}a_0^*(a_0) = q^{(1-r)(\alpha+r-1)}.$$

Now if  $\psi(\kappa, \varepsilon)$  and  $\psi(\kappa', \varepsilon')$  are conjugate in  $\overline{G}$  then  $\kappa = \kappa'$  so for any  $\lambda \in I$ , let  $\alpha = \frac{N\lambda(\sigma^2\kappa)}{2i\pi r} = \frac{N\lambda(\sigma^2\kappa')}{2i\pi r}$ , and we have  $\theta_{(\kappa, \varepsilon)}^\lambda = q^{\frac{\alpha^2 - (r-1)^2}{2}} = \theta_{(\kappa', \varepsilon')}^\lambda$ .

Finally, let  $k = (\text{Id} \otimes (C_{y, s(y)}^{\lambda, \lambda})^{\pm 1})(h \otimes \text{Id})$  then by definition of a right trace we have

$$\text{t}_{V \otimes A_y^\lambda \otimes A_{s(y)}^\lambda}(k) = \text{t}_V \left( \text{tr}_r^{A_y^\lambda \otimes A_{s(y)}^\lambda}(k) \right) = \text{t}_V \left( \text{tr}_r^{A_y^\lambda}((\text{Id}_V \otimes f)h) \right) = (\theta_y^\lambda)^{\pm 1} \text{t}_{V \otimes A_y^\lambda}(h)$$

where  $f$  is defined in Equation (26) and  $h$  is any element of  $\text{End}_{\mathcal{C}}(V \otimes A_y^\lambda)$ . Thus, we have a well defined  $G^*$ -Markov trace and the theorem is proved. 6.11

**Corollary 6.12.** *The trace coloring system given in Theorem 6.11 leads to a well defined link invariant of  $Y$ -admissible  $G$ -link, via Theorem 4.3.*

**6.7. Computation of the holonomy  $R$ -matrix for  $N = 4$ .** Here we present some computations for the elementary  $N = 4$  case. Then  $r = 2$ ,  $\xi = \mathbf{i} = \exp(i\pi/2)$ ,  $\sigma = \xi^{-\frac{r(r-1)}{2}} = -\mathbf{i}$ ,  $\{1\}_1 = 2\mathbf{i}$ . Let  $\Phi^t = \exp(tf)$ . Equation (13) implies,

$$[E, f] = F^{[1]}\{H - 1\}_1 = 2\mathbf{i}F\{H - 1\}_1, \quad [E^2, f] = \{H; 2\}_1! = -\mathbf{i}\{2H\}_1.$$

$$[E, f^n] = n2\mathbf{i}F\{H - 1\}_1 f^{n-1} \implies [E, \Phi^t] = 2t\mathbf{i}F\{H - 1\}_1 \Phi^t,$$

$$\text{Ad}_t(E) = \Phi^t E \Phi^{-t} = E - 2t\mathbf{i}F\{H - 1\}_1.$$

Choose as in Theorem 6.11 a basis  $(v_0, v_1)$  of the module  $V_\alpha^t = \Phi^{-t}V_\alpha$  whose degree is given by

$$(\kappa = -\mathbf{i}^{2\alpha} = -\exp(i\pi\alpha) \quad , \quad \varepsilon = -\mathbf{i}t\{2\alpha\} = 2t\sin(\pi\alpha)).$$

Then the actions of  $K, E, F$  in this basis are given by the matrices

$$K = \begin{pmatrix} \mathbf{i}^{\alpha+1} & 0 \\ 0 & \mathbf{i}^{\alpha-1} \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 0 & \cos \frac{\pi\alpha}{2} \\ 4t \sin \frac{\pi\alpha}{2} & 0 \end{pmatrix}$$

Remark that the matrix of  $E$  on  $V_\alpha^t$  is equal to the matrix of  $\text{Ad}_t(E)$  on  $V_\alpha^0 = V_\alpha$ .

The restriction of  $R_\xi : V_\alpha^{t_1} \otimes V_\beta^{t_2} \rightarrow V_\alpha^{t'_1} \otimes V_\beta^{t'_2}$  is given by

$$R_\xi = (\Phi^{-t'_1} \otimes \Phi^{-t'_2}) \mathbf{i}^{H \otimes H/2} (\Phi^{t_1} \otimes \Phi^{t_2}) (1 + 2\mathbf{i}E \otimes F)$$

where  $t'_1 = -\mathbf{i}^{-2\beta}t_1$  and  $t'_2 = (1 - \mathbf{i}^{-4\alpha})t_1 - \mathbf{i}^{-2\alpha}t_2$ . In the basis ordered with the lexicographic order, we have

$$R_\xi = \mathbf{i}^{\frac{\alpha\beta-1}{2}} \mathbf{i}^{-\frac{\alpha+\beta}{2}} R(\mathbf{i}^\alpha, \mathbf{i}^\beta, t_1) \text{ where } R(a, b, t) = \begin{pmatrix} \mathbf{i}ab & 0 & 0 & 0 \\ 0 & b & \mathbf{i}b(a + a^{-1}) & 0 \\ 0 & 0 & a & 0 \\ 4\mathbf{i}t(a - a^{-1}) & 0 & 0 & \mathbf{i} \end{pmatrix}$$

The holonomy braid relation implies that  $R$  satisfy the following:

$$R_{12}\left(a, b, -\frac{t_1}{c^2}\right) R_{13}(a, c, t_1) R_{23}(b, c, t_2) = \\ R_{23}\left(b, c, \frac{t_1(a^4 - 1) - t_2a^2}{a^4}\right) R_{13}\left(a, c, -\frac{t_1}{b^2}\right) R_{12}(a, b, t_1)$$

The matrices  $R$  lift the set-theoretical Yang-Baxter map

$$((a, t_1), (b, t_2)) \mapsto ((a, -\frac{t_1}{b^2}), (b, (1 - \frac{1}{a^4})t_1 - \frac{t_2}{a^2})).$$

Computations for  $N = 4, 6$  lead to the following conjecture : Let  $L$  be a link whose  $n$  components are colored by elements  $\lambda_i \in I$ . Let  $\rho : \pi_1(S^3 \setminus L) \rightarrow G$  be a representation of its group. The map  $\left( \begin{smallmatrix} \kappa & \varepsilon \\ 0 & \kappa^{-1} \end{smallmatrix} \right) \mapsto \lambda_i \left( \frac{N\lambda(\sigma^2\kappa)}{2i\pi r} \right)$  is constant on conjugacy class thus its value  $\alpha_i$  on any meridian of the  $i^{\text{th}}$  component of  $L$  is well defined. Then the semi-cyclic invariant (associated to root of unity) of  $G$ -links is given by  $F'(L, \rho) = N(L, (\alpha_1, \dots, \alpha_n))$  where  $N$  is the nilpotent (or ADO) invariant defined in [10] (see also [1, 5]) for the same root of unity.

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